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Grid-Characteristic Method Using High-Order Interpolation on Tetrahedral Hierarchical Meshes with a Multiple Time Step

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Abstract—The purpose of the present paper is to develop a grid-characteristic method for high-performance computer systems using unstructured tetrahedral hierarchical meshes, a multiple time step and the high-order interpolation for simulating complex spatial dynamic processes in heterogeneous environments. This method has the precise formulation of contact conditions and is suitable for the physically correct solution of seismology and seismic prospecting problems in complex heterogeneous environments. The use of the hierarchical meshes allows us to take into account a large number of nonhomogeneous inclusions (cracks, cavities, etc.). The use of this grid-characteristic method makes it possible to use a multiple time step and thereby increase productivity and significantly reduce the computation time. The methods developed for high-order interpolation on unstructured tetrahedral meshes can solve the problems of seismology and seismic prospecting with approximation in space of up to the fifth degree (inclusive).

Keywords: grid-characteristic method, tetrahedral grids, high-order interpolation, seismology, seismic prospecting, parallel algorithms, hierarchical grids

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1. INTRODUCTION

In modern problems for simulating spatial dynamic processes in complex heterogeneous environments, increasingly complicated mechanical mathematical models have to be introduced. Numerical experiments in seismic prospecting and seismology are among such problems.

Nowadays, seismic processing is one of the most common ways of investigating rocks before deep-hole drilling. Numerical experiments make it possible to obtain significantly refined interpretation results for seismic prospecting data and optimize the oil extraction process. In order to perform such numerical experiments, highly accurate simulation in geological environments with a large number of nonhomogeneities, such as cavities and cracks, of different shapes and with no strict patterns in their placement is required.

Simulation of earthquakes is a topical problem as well. Wave patterns obtained in the depth, at the surface, and for the objects at the surface make it possible to determine possible damage areas for housing and industrial premises and thereby increase their seismic resistance.

When this statement of spatial problems is accepted, unstructured tetrahedral meshes have to be used.

Since the system of equations for the mathematical model of the states of a continuous linearly elastic medium [1] is hyperbolic, and highly accurate calculation of wave processes is required, the grid-characteristic method [2] using high-order [4] interpolation [3] is the optimal choice. The examples of the grid-characteristic method using quadratic interpolation with a limiter on unstructured tetrahedral meshes applied to solving seismic prospecting problems may be found in [5, 6]. The examples of application of the grid-characteristic method using high-order interpolation on tetrahedral meshes may be found in [7, 8].

Transition from two-dimensional to spatial problems results in higher volumes of data. Therefore, high-performance computational systems have to be used. The developed algorithm was parallelized, and optimal use of the resources of the computational cluster was ensured.

In order to obtain thorough and detailed descriptions for all wave processes in the vicinity of all nonhomogeneities present in the problem, a rather detailed mesh has to be used. The smaller the heterogeneous inclusions to be studied, the more time steps and operations at each time layer have to be performed. However, in most cases, nonhomogeneities are localized within a small volume inside the integration domain. The use of hierarchical meshes with condensations at the locations of the nonhomogeneities is optimal for the given statement of the problem.

In grid-characteristic methods, time integration steps depend directly on the size of the minimum space steps. Thus, the use of standard hierarchical meshes does not reduce the number of time steps but only reduces the number of operations at each time layer.

The performed theoretical and numerical investigations showed that the use of grid-characteristic methods makes it possible to use dedicated hierarchical meshes with a multiple step. In addition to a multiple space step, a multiple time step may be introduced, which will reduce not only the number of operations required for integrating problems at each time layer but also the number of time steps carried out in the segment of the integration domain with no heterogeneities and with a coarser mesh.

2. STATEMENT OF THE PROBLEM

According to [1], the state of the continuous linearly elastic medium with infinitely small volume satisfies the following equations:

$$\rho \partial_t \mathbf{v} = (\nabla \cdot \boldsymbol{\sigma})^{\mathrm{T}}, \tag{1}$$

$$\partial_t \boldsymbol{\sigma} = \lambda (\nabla \cdot \mathbf{v}) \mathbf{I} + \mu \left(\nabla \otimes \mathbf{v} + (\nabla \otimes \mathbf{v})^{\mathrm{T}} \right). \tag{2}$$

Equation (1) is a local motion equation. Here, ρ is the density of the material, $\mathbf{v}$ is the motion speed, and $\boldsymbol{\sigma}$ is the Cauchy stress tensor, which is symmetric due to the pair law for shearing stresses [1]. Equation (2) is derived from Hooke's law by time differentiation. Here, λ and μ are Lame parameters, which determine the properties of the elastic material.

The following mathematical notation is used in (1), (2), and below:

$\partial_t a \equiv \dfrac{\partial a}{\partial t}$ is the partial derivative of the field a with respect to t;

$\mathbf{a} \otimes \mathbf{b}$ is the tensor product of vectors $\mathbf{a}$ and $\mathbf{b}$, $(\mathbf{a} \otimes \mathbf{b})^{ij} = a^i b^j$;

$\mathbf{I}$ is the second-rank unit tensor.

3. NUMERICAL METHOD

The grid-characteristic method on tetrahedral meshes making it possible to construct correct numerical algorithms for calculating boundary points and points lying at the interfaces of two media with different Lame parameters and (or) densities is used for the numerical solution of system (1) and (2).

Three arbitrary directions are selected as the basis at each time integration step, which ensures the method's isotropy, and new coordinates (ξ_1, ξ_2, ξ_3) are introduced. System (1), (2) may be represented in these coordinates as follows:

$$\partial_t \mathbf{q} + \mathbf{A}_1 \partial_{\xi_1} \mathbf{q} + \mathbf{A}_2 \partial_{\xi_2} \mathbf{q} + \mathbf{A}_3 \partial_{\xi_3} \mathbf{q} = 0. \tag{3}$$

In (3), vector $\mathbf{q}$ implies the vector composed of three speed components and six components of the symmetric stress tensor

$$\mathbf{q} \in \{v_1, v_2, v_3, t_{11}, t_{22}, t_{33}, t_{23}, t_{13}, t_{12}\}^{\mathrm{T}}.$$

For each of the three systems written as

$$\partial_t \mathbf{q} + \mathbf{A}_1 \partial_{\xi_1} \mathbf{q} = 0, \tag{4}$$

the following precise expression is fulfilled:

$$\mathbf{q}(\xi_1, \xi_2, \xi_3, t + \tau) = \sum_{i=1}^{I} \mathbf{X}_i \mathbf{q}(\xi_1 - c_i \tau, \xi_2, \xi_3, t). \tag{5}$$

Here, $\mathbf{X}_i$ are certain matrices expressed using the components of matrix $\mathbf{A}_1$, c_i are the eigenvalues of matrix $\mathbf{A}_1$, and τ is the time integration step.

The eigenvalues of all three matrices are expressed using density and Lame coefficients as follows:

$$c_i \in \left\{ ((\lambda + 2\mu)/\rho)^{1/2}, -((\lambda + 2\mu)/\rho)^{1/2}, (\mu/\rho)^{1/2}, -(\mu/\rho)^{1/2}, (\mu/\rho)^{1/2}, -(\mu/\rho)^{1/2}, 0, 0, 0 \right\}. \tag{6}$$

When high-order interpolation is used in (5), and formulas similar to (5), which correspond to a system similar to (4), are successively applied to directions ξ_1, ξ_2, and ξ_3, we find a way of obtaining solutions at the next time layer. Interpolation with orders ranging from the first to the fifth orders inclusively may be performed using the given software, which makes it possible to carry out numerical space integration of the solution highly accurately. In addition, the use of matrices $\mathbf{X}_i$ is implemented via two operators, which reduces the number of interpolations for each point and each direction from nine to six.

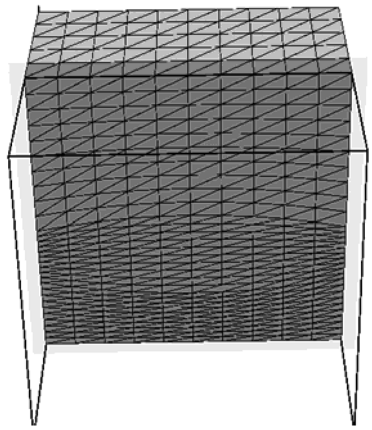

Fig. 1

4. BOUNDARY AND CONTACT CORRECTORS

The applied method makes it possible to use the most precise computational algorithms at the boundaries and contact interfaces of the integration domain.

Let us have the boundary condition written in matrix form as follows:

$$\mathbf{Dq}(\xi_1, \xi_2, \xi_3, t + \tau) = \mathbf{d}. \tag{7}$$

Here, $\mathbf{q}(\xi_1, \xi_2, \xi_3, t + \tau)$ are the values of the speed components and the stress tensor at the next integration step at the boundary point.

According to (6), there are three zero, three positive, and three negative eigenvalues for each matrix $\mathbf{A}_j$. For the sake of definiteness, we assume that the characteristics corresponding to the negative eigenvalues of matrix $\mathbf{A}_1$ are beyond the integration domain along direction ξ_1.

Then, the following is derived at the stage of calculating the inner points according to (5):

$$\mathbf{q}^{\text{in}}(\xi_1, \xi_2, \xi_3, t + \tau) = \sum_{c_i \geq 0} \mathbf{X}_i \mathbf{q}(\xi_1 - c_i\tau, \xi_2, \xi_3, t).$$

Matrix $\mathbf{\Omega}^{*,out}$ is composed of eigenvectors which correspond to negative eigenvalues.

At the boundary point, the corrector acts according to the following formula:

$$\mathbf{q}(\xi_1, \xi_2, \xi_3, t + \tau) = \mathbf{Fq}^{\text{in}}(\xi_1, \xi_2, \xi_3, t + \tau) + \mathbf{\Phi d}, \tag{8}$$

and condition (7) is fulfilled with the order equal to that of interpolation.

Matrix $\left(\mathbf{D\Omega}^{*,out}\right)^{-1}$ is obtained in (8) in such a way that

$$\left(\mathbf{D\Omega}^{*,out}\right)^{-1} \mathbf{D\Omega}^{*,out} = \mathbf{I},$$

and matrices $\mathbf{\Phi}$ and $\mathbf{F}$ are calculated using the following formulas:

$$\mathbf{\Phi} = \mathbf{\Omega}^{*,out}\left(\mathbf{D\Omega}^{*,out}\right)^{-1}, \quad \mathbf{F} = \mathbf{I} - \mathbf{\Phi D}.$$

In order to solve various problems, the boundary conditions with the prescribed external force and boundary speed, mixed boundary conditions, and nonreflecting boundary conditions based on the equality of the output characteristics to zero may be used. Equation (7) for nonreflecting boundary conditions is as follows:

$$\mathbf{\Omega}_k^{out,l/r} \mathbf{q}(t + \tau, \mathbf{r}_@) = 0.$$

The contact condition of complete adhesion is implemented using the corrector with prescribed speed. The contact condition of free slipping is calculated using the corrector for mixed boundary conditions. In this case, the speed vector and the normal speed component are calculated based on values $\mathbf{q}^{\text{in}}(\xi_1, \xi_2, \xi_3, t + \tau)$ for two contacting bodies. The example of the mesh with a parabolic layer limited by the contact surfaces is presented in Fig. 1.

5. INTERPOLATION IN A TETRAHEDRON

In order to determine a polynomial field with degree N, which depends on x, y, and z, the values at $\frac{(N+1)(N+2)(N+3)}{6}$ reference points should be known.

The following method of arranging reference points is suggested. The planes parallel to the faces of the tetrahedron $ABCD$, which divide each of its edges into N equal parts, are drawn within the tetrahedron. The reference points are numbered in the way shown in Fig. 2 for the tetrahedron, with $N = 3$.

The planes divide the tetrahedron into smaller tetrahedra similar to it and octahedra. When $N = 2$, we obtain four smaller tetrahedra and one octahedron, as shown in Fig. 3.

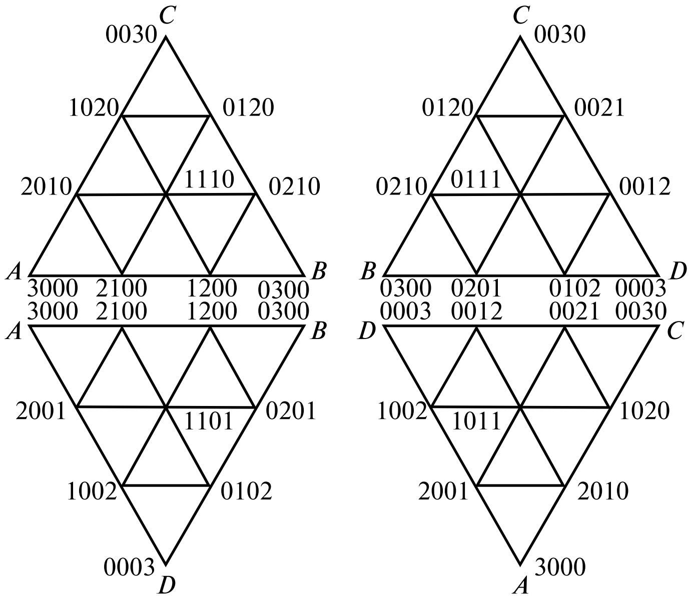

Fig. 2

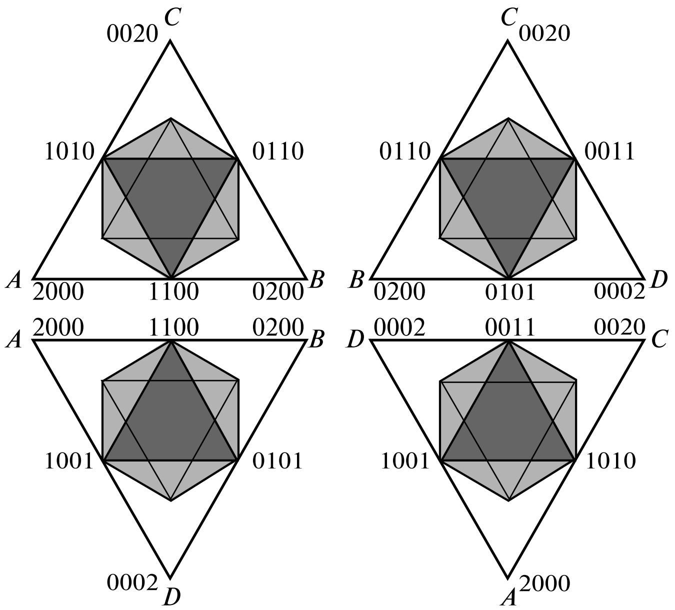

Fig. 3

The Courant number [2] is calculated based on the minimum height among all smaller tetrahedra.

We indicate the vectors of the tetrahedron vertices as $\mathbf{r}_A$, $\mathbf{r}_B$, $\mathbf{r}_C$, and $\mathbf{r}_D$. The weights of reference points $w_{abcd}(\mathbf{r})$ are calculated for each given N using the corresponding formulas. The value of the polynomial at the desired point $\mathbf{r}$ is defined using the formulas

$$v(\mathbf{r}) = \sum_{a,b,c,d} w_{abcd}(\mathbf{r})\, v_{abcd}. \tag{9}$$

In formula (9), $v_{abcd} = v(\mathbf{r}_{abcd})$ indicates the value of the interpolated function at the reference point $\mathbf{r}_{abcd}$.

The algorithm for constructing interpolants with limiters on tetrahedral meshes based on interpolation by the N-degree polynomial is as follows.

(1) The value of the trial function at the given point $\mathbf{r}$ is determined using polynomial interpolation with degree N. Let the said value be $v_N(\mathbf{r})$.

(2) We determine to which smaller tetrahedron or octahedron the point $\mathbf{r}$ belongs. If the point is within the octahedron, then the octahedron is divided by an axis into four tetrahedra, which have volumes equal to those of the other smaller tetrahedra, but are not similar to them (the axis may be drawn in three possible ways). After that, we determine to which of these four tetrahedra the point belongs.

(3) We compare $v_N(\mathbf{r})$ to the minimum m and the maximum M of the values at the vertices of the tetrahedron:

(3.1) If $m \le v_N(\mathbf{r}) \le M$, then the interpolant value at $\mathbf{r}$ is $v_N(\mathbf{r})$;

(3.2) If $v_N(\mathbf{r}) < m$, then the interpolant value at $\mathbf{r}$ is m;

(3.3) If $v_N(\mathbf{r}) > M$, then the interpolant value at $\mathbf{r}$ is M.

The use of interpolation with a limiter makes it possible to eliminate nonphysical oscillations of polynomials, which occur in the presence of discontinuities in the interpolated polynomial functions.

6. HIERARCHICAL MESHES WITH MULTIPLE TIME STEP

The principles of using hierarchical meshes with a multiple step will be explained below for the multiplicity of two using the following system of equations composed of two transfer equations as an example:

$$u_t + u_x = 0, \tag{10}$$

$$v_t - v_x = 0. \tag{11}$$

This system of equations (10) and (11) is hyperbolic. Therefore, the following expressions are fulfilled for its solution:

$$u(t + \tau, x_@) = u(t, x_@ - \tau), \tag{12}$$

$$v(t + \tau, x_@) = v(t, x_@ + \tau), \tag{13}$$

$$u(t+2\tau, x_@) = u(t+\tau, x_@ - \tau), \quad (14)$$

$$v(t+2\tau, x_@) = v(t, x_@ + 2\tau). \quad (15)$$

In (12)–(15), $x_@$ indicates the coordinate, to the left of which the mesh is fine and to the right of which it is coarse (Fig. 4), t is the current time, and τ is the time integration step in the domain with the fine mesh. Correspondingly, the step in the domain with the coarse mesh will be 2τ.

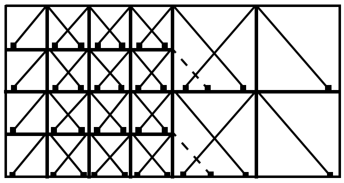

Fig. 4

In the course of solving the system of state equations for a linearly elastic medium, two ways of calculating boundary points with coordinate $x_@$ may be considered using the following principle: the values of the characteristics are calculated at various points depending on time and the mesh size in the domain to which the characteristic belongs and summed; or the corrector based on the contact corrector of complete adhesion is used at the boundary. It was found in the process of mathematical and numerical investigations that the generalization of the second approach for salient points is better, and nonphysical oscillations do not arise at these points when this approach is used.

The example of a hierarchical tetrahedral grid with a multiplicity of two is presented in Fig. 5. The result of numerical simulation of the passage of a seismic wave through such a grid is presented in Fig. 6; the calculation is performed with multiple time step. The second-order polynomial interpolation was used at the reference points for the fourth order without a limiter. The speed modulus corresponds to the density of the visualized environment. It can be seen that no nonphysical oscillations arise, even when such a weak interpolator is used.

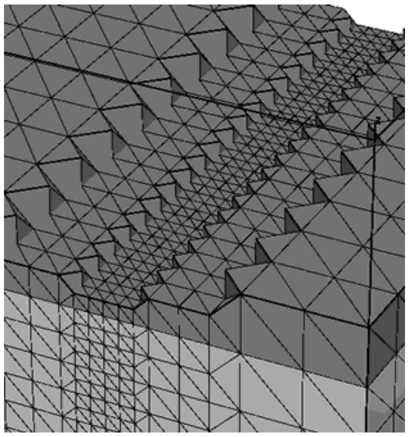

Fig. 5

6. RESULTS

The following six problems are considered:

(1) Numerical simulation of a frontal collision with a face of a cube.

(2) Numerical simulation of a spherical explosion at the center of a cube.

(3) Numerical simulation of a near-surface seismic spherical explosion in a linearly elastic medium.

(4) Numerical simulation of an earthquake in the Earth's crust.

(5) Numerical simulation of the passage of a seismic wave through the interface of two media having a parabolic shape.

(6) Numerical simulation of the passage of a seismic wave through the layer with different elastic parameters and with parabolic boundaries.

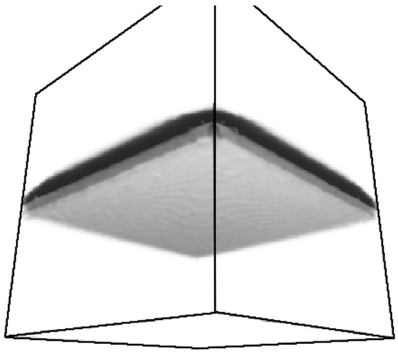

Fig. 6

The integration domain in all six cases is a cube in which the unstructured tetrahedral mesh required for each particular problem is generated. The free boundary condition is set on all sides of the cube. The speed modulus corresponds to the density of the visualized environment in all the figures.

Interpolation with a limiter on a cubic basis was used in the first four problems. Second-order interpolation with a limiter was used in the fifth problem, and second-order interpolation on fourth-order points without a limiter was used in the sixth problem.

The result of the numerical simulation of a frontal collision with the center of a wall of a cube is presented in Fig. 7, and the result for a spherical explosion at the center of a cube is presented in Fig. 8. In both problems, the mesh is composed of about 200 000 tetrahedra not divided into auxiliary ones; nearly 100 time steps were carried out.

The result of the simulation of a near-surface seismic explosion is presented in Fig. 9, and the result of earthquake simulation is presented in Fig. 10. In these two problems, the mesh is composed of about 500 000 undivided tetrahedra. In order to simulate the near-surface explosion, 1200 time steps were performed, and 700 steps were performed for the earthquake simulation.

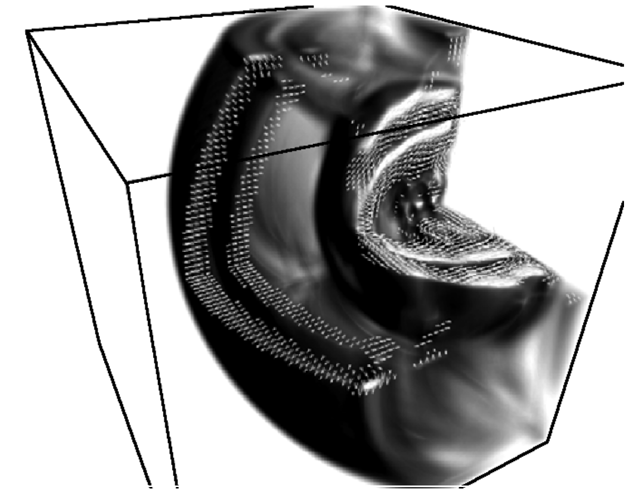

Fig. 7

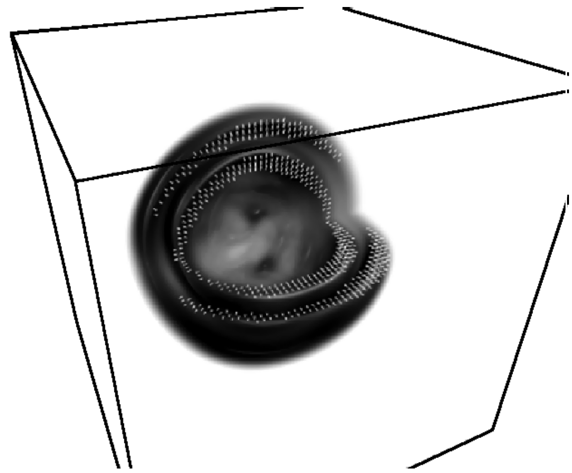

Fig. 8

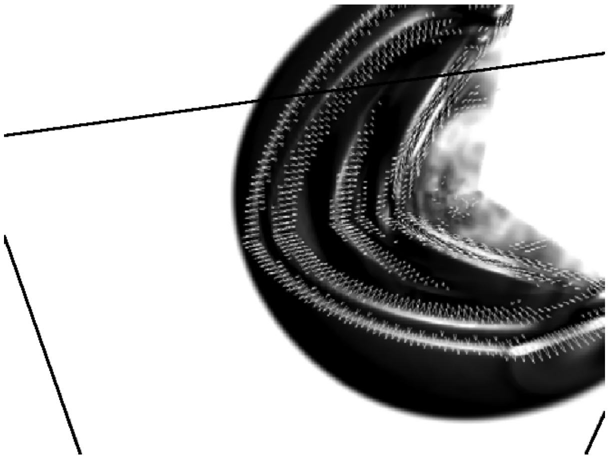

Fig. 9

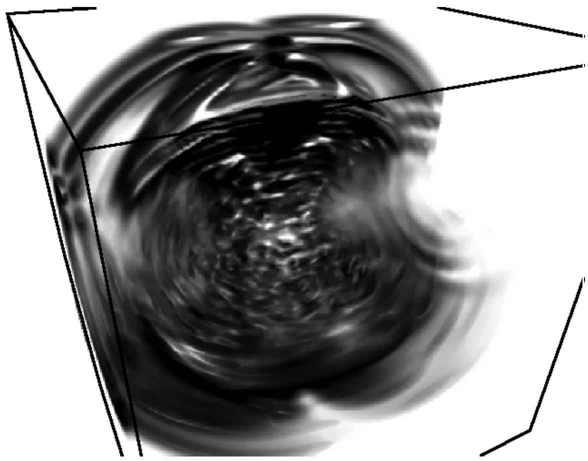

Fig. 10

The pattern of arising waves may be seen in the first four problems. These are bulk seismic longitudinal and transverse waves, as well as surface seismic Rayleigh waves. The white vectors indicate the direction of speed.

The result of numerical simulation of the passage of a seismic wave through the interface of two media having a parabolic shape is presented in Fig. 11. The edge of the seismic cube is 200 m. In this problem, the mesh consists of 64 000 nodes. About 400 time steps were performed, which corresponds to 1.5 seconds. The parameters of the environment above the interface are as follows: the speed of the longitudinal seismic waves is $c_p = 4230$ m/s, the speed of transverse seismic waves is $c_s = 3000$ m/s, and the density of the environment is $\rho = 2400$ kg/m^3. The parameters below the interface are $c_p = 2115$ m/s, $c_s = 1500$ m/s, and $\rho = 2400$ kg/m^3. The seismic wave reflected from the parabolic layer can be seen in Fig. 11, and the Berlage pulse distorted after passing through the interface of two media having a parabolic shape may be seen above the wave.

The result of the numerical simulation of the passage of a seismic wave through a layer with different elastic parameters and with parabolic boundaries is presented in Fig. 12. The mesh presented in Fig. 1 consists of about 70 000 nodes. Two reflected waves can be seen in Fig. 12. The first wave (the upper one in the figure) was reflected from the upper boundary of the interface between two media, and the second one was reflected from the lower boundary. In addition, the distortion of the shape of the wave that passed through the layer can be seen (the lowest of the three in the figure).

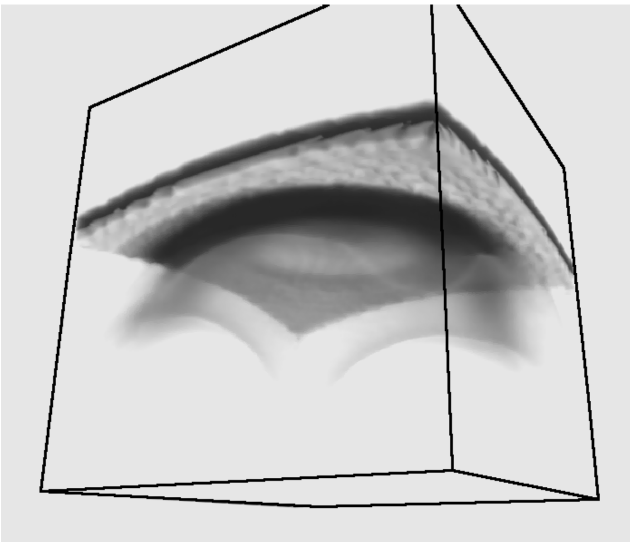

Fig. 11

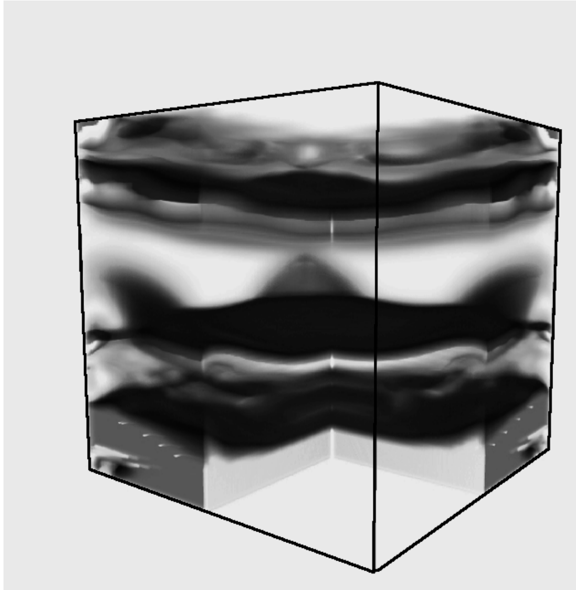

Fig. 12

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