IAA-AAS-DyCoSS3-038
AAS 17-851

SATELLITE DYNAMICS UNDER THE INFLUENCE OF GRAVITATIONAL AND DAMPING TORQUES

Vasily A. Sarychev* and Sergey A. Gutnik†‡

In this paper the dynamics of the rotational motion of a satellite, moving in the central Newtonian force field in a circular orbit under the influence of gravitational and active damping torques, depending on the projections of the angular velocity of the satellite is investigated. The main attention is given to study the necessary and sufficient conditions for asymptotic stability of satellite's equilibria for special case, when the principal axes of inertia of the satellite coincide with the axes of the orbital coordinate system.

INTRODUCTION

The study of satellite dynamics under the influence of gravitational and active damping torques is an important topic of orientation control systems. The gravity orientation systems are based on the result that a satellite with different moments of inertia in the central Newtonian force field in a circular orbit has 24 equilibrium orientations and four of them are stable.[1,2] An important property of gravity orientation systems is that these systems can operate for a long time without spending energy. The problem to be analyzed in the present work is related to the motion of the satellite acted upon by the gravity gradient and active damping torques. We assume that active damping torques depend on the projections of the angular velocity of the satellite. Such active damping torques can be provided by using the angular velocity sensor.

The action of damping torques both leads to new equilibrium orientations and can provide the asymptotic stability of the well-known equilibria of the gravity oriented satellites. Therefore, it is important to study the joint action of gravitational and active damping torques and, in particular, to analyze necessary and sufficient conditions for asymptotic stability of satellite's equilibria in a circular orbit. Such solutions can be used in practical space technology in the design of control systems of orientation of the satellites.

In this paper, the problem of determination of the conditions for asymptotic stability of equilibria for the general values of damping torques is considered. The conditions of equilibria stability are determined as a result of analysis of the linearized equations of motion using Routh-Hurwitz criterion. The detailed investigation of the regions of the necessary and sufficient conditions of stability is studied by a numerical-analytical method in the plane of two dimensionless

* Prof., Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Miusskaya pl. 4, Moscow, 125047, Russia. E-mail: vas31@rambler.ru.
† Dr., Department of Mathematics and IT, Moscow Institute of International Relations, pr. Vernadskogo 76, Moscow, 119454, Russia. E-mail: s.gutnik@inno.mgimo.ru.
‡ Dr., Department of Informatics and Computational Mathematics, Moscow Institute of Physics and Technology, Institutskii per. 9, Dolgoprudnyi, Moscow Region, 141700, Russia.

inertia parameters at different values of damping coefficients. The types of transition decay processes of a satellite at different damping parameters have been investigated numerically.

EQUATIONS OF MOTION

Consider the attitude motion of a satellite-rigid body subjected to gravitational and active damping torques in a circular orbit. We assume that active damping torques depend on the projections of the angular velocity of the satellite. To write the equations of motion we introduce two right-handed Cartesian coordinate systems with origin in the satellite's center of mass O. The orbital reference frame $OXYZ$; the axis OZ is directed along the radius vector from the Earth center of mass to the satellite's center of mass; the axis OX is in the direction of a satellite's orbital motion. The satellite's body reference frame $Oxyz$; Ox, Oy, Oz – are the principal central axes of inertia of the satellite. The orientation of the satellite body coordinate system $Oxyz$ with respect to the orbital coordinate system is determined by means of the aircraft angles α, β and γ. The direction cosines in transformation matrix between the orbital coordinate system and the satellite's body reference frame $Oxyz$ are represented by the following expressions[1]

$$\begin{aligned}
a_{11} &= \cos(x,X) = \cos\alpha\cos\beta,\\
a_{12} &= \cos(y,X) = \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma,\\
a_{13} &= \cos(z,X) = \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma,\\
a_{21} &= \cos(x,Y) = \sin\beta,\\
a_{22} &= \cos(y,Y) = \cos\beta\cos\gamma,\\
a_{23} &= \cos(z,Y) = -\cos\beta\sin\gamma,\\
a_{31} &= \cos(x,Z) = -\sin\alpha\cos\beta,\\
a_{32} &= \cos(y,Z) = \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma,\\
a_{33} &= \cos(z,Z) = \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma.
\end{aligned} \tag{1}$$

For small oscillations of the satellite the angles of pitch, yaw, and roll correspond to rotations around the OY, OZ and OX axes, respectively.

Let the active damping moments act on the satellite. The summary vector projections of these moments on the axis Ox, Oy, Oz are equal to the following values $M_x = \bar{k}_1 p_1$, $M_y = \bar{k}_2(q_1 - \omega_0)$, $M_z = \bar{k}_3 r_1$ respectively. Here $\bar{k}_1$, $\bar{k}_2$, $\bar{k}_3$ – are the damping coefficients, p_1, q_1, r_1 – are the projections of the satellite's angular velocity onto the axes Ox, Oy, Oz; ω_0 – is the angular velocity of the orbital motion of the satellite's center of mass. Then equations of the satellite's attitude motion can be written in the Euler form

$$\begin{aligned}
&Ap_1' + (C-B)q_1 r_1 - 3\omega_0^2(C-B)a_{32}a_{33} + \bar{k}_1 p_1 = 0,\\
&B q_1' + (A-C)r_1 p_1 - 3\omega_0^2(A-C)a_{33}a_{31} + \bar{k}_2(q_1 - \omega_0) = 0,\\
&C r_1' + (B-A)p_1 q_1 - 3\omega_0^2(B-A)a_{31}a_{32} + \bar{k}_3 r_1 = 0;
\end{aligned} \tag{2}$$

$$p_1 = (\alpha' + \omega_0)a_{21} + \gamma',$$
$$q_1 = (\alpha' + \omega_0)a_{22} + \beta' \sin\gamma, \tag{3}$$
$$r_1 = (\alpha' + \omega_0)a_{23} + \beta' \cos\gamma.$$

In Eqs. (2) – (3) A, B, C – are the principal central moments of inertia of the satellite. The symbol "prime" denotes differentiation with respect to time t.

Introducing dimensionless parameters $\theta_A = A/B,\ \theta_C = C/B,\ p = p_1/\omega_0,\ q = q_1/\omega_0,$ $r = r_1/\omega_0,\ k_1 = \overline{k}_1/\omega_0 B,\ k_2 = \overline{k}_2/\omega_0 B,\ k_3 = \overline{k}_3/\omega_0 B,\ \tau = \omega_0 t$ system (2) – (3) takes the form

$$\theta_A \dot{p} + (\theta_C - 1)qr - 3(\theta_C - 1)a_{32}a_{33} + k_1 p = 0,$$
$$\dot{q} + (\theta_A - \theta_C)rp - 3(\theta_A - \theta_C)a_{33}a_{31} + k_2(q-1) = 0, \tag{4}$$
$$\theta_c \dot{r} + (1 - \theta_A)pq - 3(1 - \theta_A)a_{31}a_{32} + k_3 r_1 = 0;$$

$$p = (\dot{\alpha} + 1)a_{21} + \dot{\gamma},$$
$$q = (\dot{\alpha} + 1)a_{22} + \dot{\beta}\sin\gamma, \tag{5}$$
$$r = (\dot{\alpha} + 1)a_{23} + \dot{\beta}\cos\gamma.$$

Putting in Eqs. (2) – (3) $\alpha = \alpha_0 = \text{const},\ \beta = \beta_0 = \text{const},\ \gamma = \gamma_0 = \text{const}$, we obtain at $A \neq B \neq C$ the equations

$$a_{22}a_{23} - 3a_{32}a_{33} + k_1 a_{21} = 0,$$
$$a_{23}a_{21} - 3a_{33}a_{31} + k_2(a_{22} - 1) = 0, \tag{6}$$
$$a_{21}a_{22} - 3a_{31}a_{32} + k_3 a_{23} = 0,$$

which specifies with the orthogonality conditions for the direction cosines the equilibrium orientations of the satellite in the orbital coordinate system.

The main focus of this work is given to study the necessary and sufficient conditions for asymptotic stability of satellite's equilibria for special case, when the principal axes of inertia of the satellite coincide with the axes of the orbital coordinate system:

$$\alpha_0 = \beta_0 = \gamma_0 = 0. \tag{7}$$

NECESSARY AND SUFFCIENT CONDITIONS OF ASYMPTOTIC STABILITY OF THE EQUILIBRIUM ORIENTATIONS OF A SATELLITE

In order to study the necessary and sufficient conditions of asymptotic stability of the equilibrium orientation (7) let us linearize system of equations (4) and (5) in the vicinity of the equilibrium solution $\alpha = \alpha_0,\ \beta = \beta_0,\ \gamma = \gamma_0$ We represent α, β, γ in the form $\alpha = \alpha_0 + \overline{\alpha},\ \beta = \beta_0 + \overline{\beta},\ \gamma = \gamma_0 + \overline{\gamma}$, where $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ – are small deviations from the equilibrium orientation of the satellite $\alpha = \alpha_0,\ \beta = \beta_0,\ \gamma = \gamma_0$. Then the linearized system of equations of motion (4) and (5) takes the following form:

$$
\begin{aligned}
&\theta_A \ddot{\bar{\alpha}} \sin\beta_0 + [2(\theta_C - 1) a_{22} a_{23} + k_1 a_{21}] \dot{\bar{\alpha}} + 3(\theta_C - 1)(a_{12} a_{33} + a_{13} a_{32}) \bar{\alpha} + \\
&+ \cos\beta_0 [(\theta_A + \theta_C - 1) - 2(\theta_C - 1)\sin^2\gamma_0] \dot{\bar{\beta}} + \\
&+ \cos\beta_0 \{(\theta_C - 1)[(1 + 3\sin^2\alpha_0)\sin\beta_0 \sin 2\gamma_0 - \frac{3}{2}\sin 2\alpha_0 \cos 2\gamma_0] + k_1\} \bar{\beta} + \\
&+ \theta_A \ddot{\bar{\gamma}} + k_1 \dot{\bar{\gamma}} + (\theta_C - 1)[(a_{23}^2 - a_{22}^2) - 3(a_{33}^2 - a_{32}^2)] \bar{\gamma} = 0, \\
&\ddot{\bar{\alpha}} a_{22} + [2(\theta_A - \theta_C) a_{21} a_{23} + k_2 a_{22}] \dot{\bar{\alpha}} + 3(\theta_A - \theta_C)(a_{13} a_{31} + a_{11} a_{33}) \bar{\alpha} + \\
&+ \ddot{\bar{\beta}} \sin\gamma_0 + [(\theta_A - \theta_C - 1)\sin\beta_0 \cos\gamma_0 + k_2 \sin\gamma_0] \dot{\bar{\beta}} - \\
&- \{(\theta_A - \theta_C)[(1 + 3\sin^2\alpha_0)\cos 2\beta_0 \sin\gamma_0 + \frac{3}{2}\sin 2\alpha_0 \sin\beta_0 \cos\gamma_0] + \\
&+ k_2 \sin\beta_0 \cos\gamma_0\} \bar{\beta} + (\theta_A - \theta_C + 1) \dot{\bar{\gamma}} a_{23} + [(\theta_C - \theta_A)(a_{21} a_{22} - 3 a_{31} a_{32}) + k_2 a_{23}] \bar{\gamma} = 0, \\
&\theta_C \ddot{\bar{\alpha}} a_{23} + \cos\beta_0 [2(1 - \theta_A)\sin\beta_0 \cos\gamma_0 - k_3 \sin\gamma_0] \dot{\bar{\alpha}} + 3(1 - \theta_A)(a_{11} a_{32} + a_{12} a_{31}) \bar{\alpha} + \\
&+ \theta_C \ddot{\bar{\beta}} \cos\gamma_0 + [(\theta_C - \theta_A + 1)\sin\beta_0 \sin\gamma_0 + k_3 \cos\gamma_0] \dot{\bar{\beta}} + \\
&+ \{(1 - \theta_A)[(1 + 3\sin^2\alpha_0)\cos 2\beta_0 \cos\gamma_0 - \frac{3}{2}\sin 2\alpha_0 \sin\beta_0 \sin\gamma_0] + \\
&+ k_3 \sin\beta_0 \sin\gamma_0\} \bar{\beta} - (\theta_A + \theta_C - 1) \dot{\bar{\gamma}} a_{22} + [(1 - \theta_A)(a_{21} a_{23} - 3 a_{31} a_{33}) - k_3 a_{22}] \bar{\gamma} = 0.
\end{aligned}
\tag{8}
$$

Now let us consider small oscillations of the satellite in the vicinity of the specific equilibrium orientation (7), when the principal axes of inertia of the satellite coincide with the axes of the orbital coordinate system: Taking into account expressions (1) for solution (7) we get $\sin\alpha_0 = 0,\ \sin\beta = 0_0,\ \sin\gamma_0 = 0$ and linearized equations (8) take the form

$$
\begin{aligned}
&\ddot{\bar{\alpha}} + k_2 \dot{\bar{\alpha}} + 3(\theta_A - \theta_C) \bar{\alpha} = 0, \\
&\theta_C \ddot{\bar{\beta}} + k_3 \dot{\bar{\beta}} - (\theta_A + \theta_C - 1) \dot{\bar{\gamma}} + (1 - \theta_A) \bar{\beta} - k_3 \bar{\gamma} = 0, \\
&\theta_A \ddot{\bar{\gamma}} + (\theta_A + \theta_C - 1) \dot{\bar{\beta}} + k_1 \dot{\bar{\gamma}} + k_1 \bar{\beta} + 4(1 - \theta_C) \bar{\gamma} = 0.
\end{aligned}
\tag{9}
$$

The characteristic equation of system (9)

$$
[\lambda^2 + k_2 \lambda + 3(\theta_A - \theta_C)](A_0 \lambda^4 + A_1 \lambda^3 + A_2 \lambda^2 + A_3 \lambda + A_4) = 0,
\tag{10}
$$

decomposes into quadratic and 4th degree equations. Here the following designations in (10) are introduced:

$$
\begin{aligned}
&A_0 = \theta_A \theta_C, \quad A_1 = k_1 \theta_C + k_3 \theta_A, \\
&A_2 = k_1 k_3 + (\theta_A + \theta_C - 1)^2 + \theta_A (1 - \theta_A) + 4\theta_C (1 - \theta_C), \\
&A_3 = k_1 \theta_C + k_3 (3 + \theta_A - 3\theta_C), \quad A_4 = k_1 k_3 + 4(1 - \theta_A)(1 - \theta_C).
\end{aligned}
$$

The necessary and sufficient conditions for asymptotic stability (Routh-Hurwitz criterion) of the equilibrium solution (7) take the following form:

$$
\begin{aligned}
&k_2 > 0, \quad \theta_A - \theta_C > 0, \\
&\Delta_1 = A_1 = k_1\theta_C + k_3\theta_A > 0, \\
&\Delta_2 = A_1A_2 - A_0A_3 = k_1^2k_3\theta_C + k_1k_3^2\theta_A + \\
&+(1-\theta_C)[k_1\theta_C(1-\theta_A+3\theta_C) + k_3\theta_A(1-\theta_A)] > 0, \\
&\Delta_3 = A_1A_2A_3 - A_0A_3^2 - A_1^2A_4 = 3(1-\theta_C)\{k_1^2k_3^2\theta_C + k_1k_3^3\theta_A + \\
&+k_1^2\theta_C^2(\theta_A+\theta_C-1) + k_1k_3\theta_C[(\theta_A+\theta_C-1)(2\theta_A-1) + 3\theta_C(1-\theta_C)] - \\
&-k_3^2\theta_A(1-\theta_A)(\theta_A+\theta_C-1)\} > 0, \\
&\Delta_4 = \Delta_3A_4 > 0, \quad A_4 = k_1k_3 + 4(1-\theta_A)(1-\theta_C) > 0.
\end{aligned}
\tag{11}
$$

Let us consider the special case, when $k_1 = k_2 = k_3 = k$. In this case, conditions (11) take more simple form

$$
\begin{aligned}
&k > 0, \quad \theta_A - \theta_C > 0, \\
&\Delta_1 = k(\theta_C + \theta_A) > 0, \\
&\Delta_2 = k[k^2 + (1-\theta_C)^2]\theta_A + k[k^2 + (1-\theta_C)(1+3\theta_C)]\theta_C - k(1-\theta_C)\theta_A^2 > 0, \\
&\Delta_3 = 3k^2(1-\theta_C)\{\theta_A^3 + (3\theta_C-2)\theta_A^2 + [k^2 + (1-\theta_C)(1-3\theta_C)]\theta_A + \\
&+\theta_C[k^2 + (1-\theta_C)(1+2\theta_C)]\} > 0, \\
&\Delta_4 = \Delta_3A_4 > 0, \quad A_4 = k^2 + 4(1-\theta_A)(1-\theta_C) > 0.
\end{aligned}
\tag{12}
$$

The detailed analysis of the regions, where necessary and sufficient conditions of stability (12) hold is studied in the plane of two dimensionless inertia parameters (θ_A, θ_C) at different values of damping coefficient k. It should be noted that along with (12) the triangle inequalities should also be satisfied $1+\theta_A \ge \theta_C$, $1+\theta_C \ge \theta_A$, $\theta_A + \theta_C \ge 1$. One may disregard the first triangle inequality, since when $\theta_A > \theta_C$ it holds automatically. Thus, the region is limited by the straight lines

$$
\theta_C = 1-\theta_A, \quad \theta_C = \theta_A, \quad \theta_C = \theta_A - 1. \tag{13}
$$

An example of such region and also all the lines on which one of inequalities (13) converts into equality are shown in Figures 1 - 4. The region where the necessary and sufficient conditions of stability are satisfied is marked out by gray color.

For small values of k close to zero (Figure 1, $k = 0.01$) the region of stability approaches to the region where the necessary and sufficient stability conditions for the satellite –rigid body for $k = 0$ takes place[1]. This region is bounded by the lines $\theta_C = 1-\theta_A$, $\theta_C = \theta_A$, $\theta_A = 1$.

In Figure 2 the region of fulfillment of the necessary and sufficient conditions of stability (12) for $k = 0.5$ is bounded by the straight lines (13) and by hyperbola $A_4 = 0$. In Figure 3 for $k = 1.0$ and Figure 4 for $k = 1.1$ the region, where conditions of stability (12) hold, is bounded only by the straight lines (13).

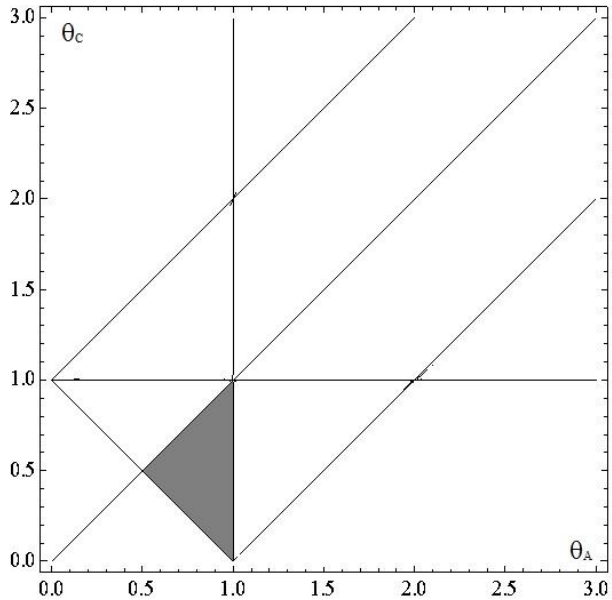

Figure 1. The domain of the conditions of asymptotic stability (grey) for k=0.01**.**

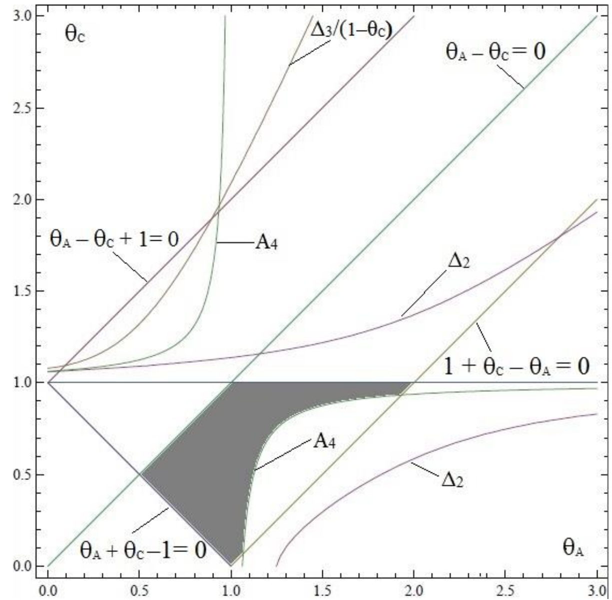

Figure 2. The domain of the conditions of asymptotic stability (grey) for k=0.5.

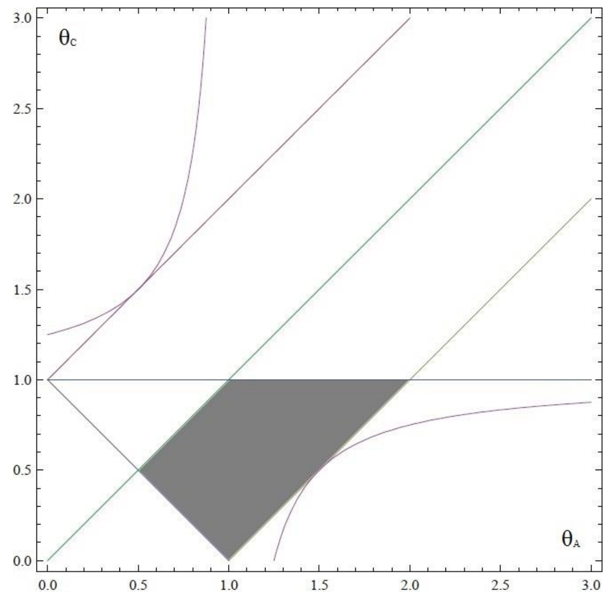

Figure 3. The domain of the conditions of asymptotic stability (grey) for k=1.0.

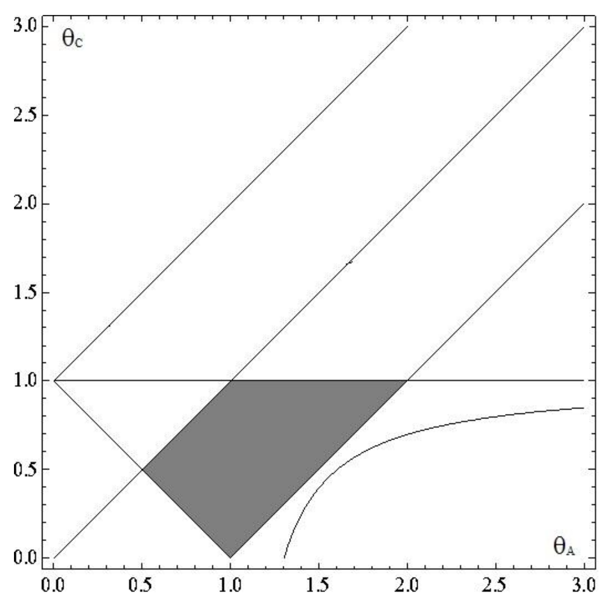

Figure 4. The domain of the conditions of asymptotic stability (grey) for k=1.1.

ANALYSIS OF TRANSITION DECAY PROCESSES

Let us now analyze the transient processes described by system of differential equations (4), (5) for various damping coefficients. The numerical integration of system (4) and (5) has been done in special case, when $k_1 = k_2 = k_3 = k$. The integration was carried out for fixed values of parameters satisfying for conditions (12).

Figure 5 shows an example of transition decay processes of spatial oscillations for $k = 0.2$ and $\theta_A = 1$, $\theta_C = 0.5$ where conditions of asymptotic stability (12) hold. It should be noted that in this case the process of the system transition to the zero equilibrium orientation along the angles β and γ is faster than in the angle α . With the value of the damping coefficient $k = 0.5$ the process of the system transition to the zero equilibrium orientation occurs about twice as fast as when $k = 0.2$. (Figure 6)

With the parameter k equal to 1 (Figure 7), the rate of the transient process increases, and the system in this case goes to the zero equilibrium orientation at all three angles for the value $\tau = 10$, which approximately corresponds to two turnovers of the satellite in the orbit. At $k = 1.5$ (Figure 8) and with a further increase of the parameter k , the nature of the transient processes varies insignificantly.

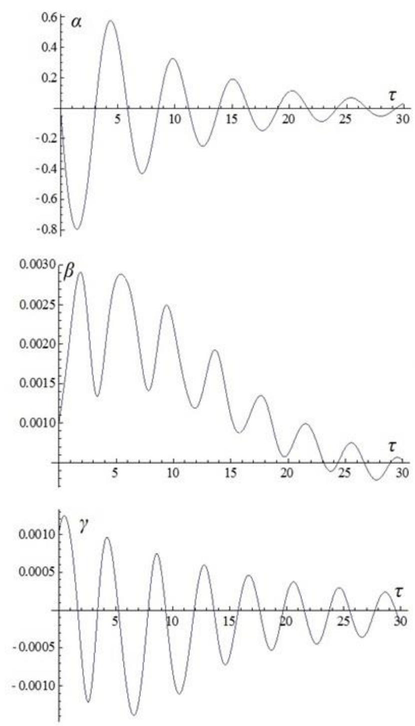

Figure 5. The transition decay processes for θ_A=1, θ_C=0.5, k=0.2

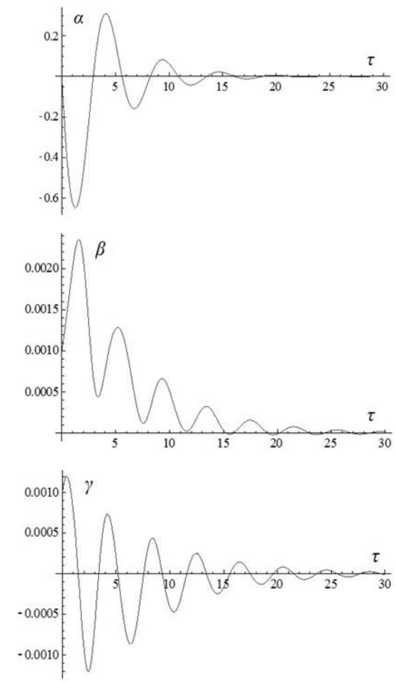

Figure 6. The transition decay processes for θ_A=1, θ_C=0.5, k=0.5

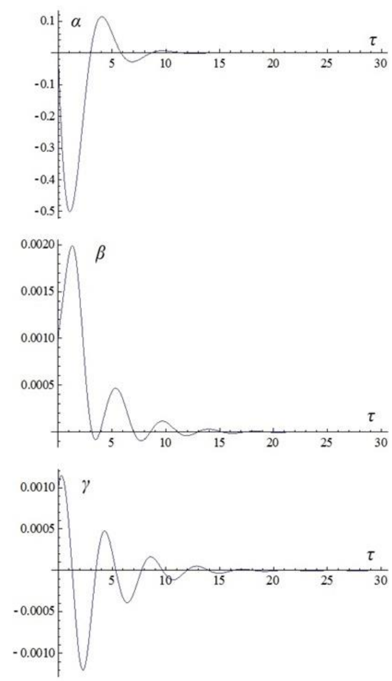

Figure 7. The transition decay processes for θ_A=1, θ_C=0.5, k=1.0

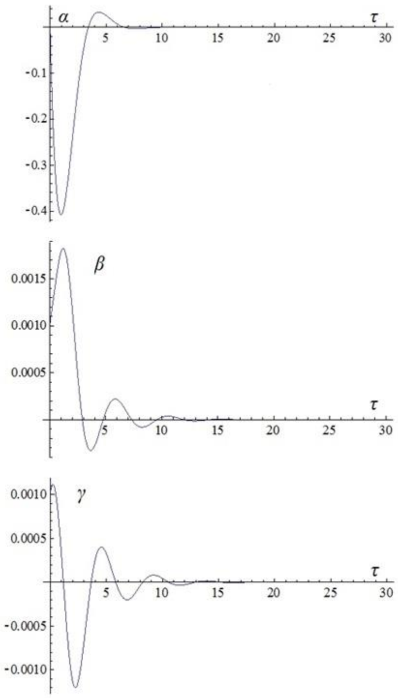

Figure 8. The transition decay processes for θ_A=1, θ_C=0.5, k=1.5

CONCLUSION

In this paper, we have analyzed the rotational motion of the satellite relative to the center of mass in a circular orbit due to gravitational and active damping torques. The main focus is the investigation of the stability conditions of the satellite equilibrium orientations. Necessary and sufficient conditions for asymptotic stability of the equilibrium orientations were obtained with the help of the Routh-Hurwitz criterion. The transition decay processes of spatial oscillations of the satellite have been investigated numerically.

The results of the study can be used in practical space technology in the design of attitude control systems of satellites.

REFERENCES

[1] V.A. Sarychev, Problems of orientation of satellites, Itogi Nauki i Tekhniki, Ser. "Space Research", Vol. 11. Moscow: VINITI, 1978.

[2] V.V. Beletsky, Attitude Motion of Satellite in Gravitational Field. Moscow: MGU Press, 1975.