ISSN 1064-2307, Journal of Computer and Systems Sciences International, 2015, Vol. 54, No. 3, pp. 469–482. © Pleiades Publishing, Ltd., 2015.
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CONTROL SYSTEMS OF MOVING OBJECTS

Dynamics of a Gyrostat Satellite Subjected to the Action of Gravity Moment. Equilibrium Attitudes and Their Stability

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Received December 2, 2014; in final form, December 9, 2014

Abstract—We study the dynamics of the rotational motion of a gyrostat satellite moving in the central Newtonian force field along a circular orbit. We propose a method for determining the equilibrium attitudes (equilibrium orientations) of a gyrostat satellite in the orbital coordinate system for given values of the gyrostatic moment vector and principal central moments of inertia, and obtain their existence conditions. For each equilibrium orientation, sufficient conditions for stability are obtained using a generalized energy integral such as a Lyapunov function. We conduct a detailed numerical analysis of domains where the stability conditions for equilibrium attitudes are satisfied depending on four dimensionless parameters of the problem. It is shown that the number of equilibrium attitudes of a gyrostat satellite for which the sufficient conditions of stability are satisfied in the general case varies from four to two with an increase in the magnitude of a gyrostatic moment. The results obtained in this paper can be used for constructing gravitational systems of control over the orientation of the Earth's artificial satellites.

DOI: 10.1134/S1064230715030107

INTRODUCTION

The orientation of an artificial satellite of the Earth can be performed by both active and passive methods. In the development of passive systems for satellite orientation, one can use the properties of gravitational and magnetic fields, the atmospheric drag effect, solar radiation pressure, and the gyroscopic properties of rotating bodies. An important feature of passive systems for satellite orientation is that they can operate in orbit for a long time without spending energy and (or) a working body. Among passive orientation systems, the most widespread are the gravitational systems which operate based on the fact that in the central Newtonian force field a satellite with unequal principal central moments of inertia has 24 equilibrium attitudes on a circular orbit, four of which are stable [1–3]. The use of rotors rotating with a constant angular velocity relative to the satellite body makes it possible to obtain new and more complex equilibrium attitudes of the gyrostat satellite that are of interest for practical applications.

The problem of determining the equilibrium attitudes of a gyrostat satellite has been addressed in many studies. The dynamics of satellites with gravitational orientation systems is considered in detail in [4]. In [5–9], the equilibrium attitudes of a gyrostat satellite and their stability were investigated for the special cases when the vector of the gyrostatic moment is parallel to one of the principal central axes of inertia of the gyrostat satellite or is located in one of the planes formed by the principal central axes of inertia. The equilibrium attitudes and their stability for an axisymmetric gyrostat satellite were considered in [10].

The general case of the problem was first considered in [11] presenting the theoretical results of a symbolic–numerical study of equilibrium attitudes of a gyrostat satellite subjected to the action of gravity and gyrostatic moments. It was shown that there are no more than 24 equilibrium attitudes in the orbital coordinate system on a circular orbit for a gyrostat satellite with a given vector of gyrostatic moment and given principal central moments of inertia.

In [12], a method based on algorithms for constructing Gröbner bases and on the concept of the resultant was used to investigate the equilibrium attitudes of a gyrostat satellite, determine the bifurcation values of system parameters for which the number of equilibrium attitudes changes, and perform a detailed numerical analysis of the evolution of the domains with a different number of equilibrium attitudes in the space of parameters.

Section 1 of this paper formulates the problem and describes the equations of rotational motion of a gyrostat satellite under the action of gravity moment. Section 2 considers the symbolic–numerical method for determining the equilibrium attitudes of a gyrostat satellite. Section 3 investigates the equilibrium attitudes of a gyrostat satellite for the given values of the problem parameters. Section 4 presents the results of the study of sufficient conditions for the stability of the equilibrium attitudes of a gyrostat satellite depending on four dimensionless parameters of the problem, using a generalized integral of energy as a Lyapunov function.

1. EQUATIONS OF MOTION

We consider the motion of a gyrostat satellite (hereafter, referred also as satellite or gyrostat) as a solid body with statically and dynamically balanced rotors located inside it. We assume that the angular velocities of rotor rotation relative to the satellite body are constant and the center of mass of the gyrostat satellite moves along a circular orbit. To write the equations of motion, we introduce two right-hand Cartesian coordinate systems with their origin located at the center of mass O of the gyrostat satellite.

$OXYZ$ is the orbital coordinate system. The OZ-axis is directed along the radius-vector connecting the centers of mass of the Earth and the satellite; the OX-axis is directed along the vector of linear velocity of the center of mass O of the satellite.

$Oxyz$ is the coordinate system bound with the satellite; Ox, Oy, Oz are the principal central axes of inertia of the satellite.

We determine the orientation of the coordinate system $Oxyz$ relative to the orbital coordinate system based on Euler angles ψ, ϑ, φ. The direction cosines of the axes Ox, Oy, Oz in the orbital coordinate system are expressed through the classical Euler angles by the relations [4]:

	x	y	z
X	a_{11}	a_{12}	a_{13}
Y	a_{21}	a_{22}	a_{23}
Z	a_{31}	a_{32}	a_{33},

$$\begin{aligned}
a_{11} &= \cos\psi\cos\varphi - \sin\psi\cos\vartheta\sin\varphi,\\
a_{12} &= -\cos\psi\sin\varphi - \sin\psi\cos\vartheta\cos\varphi,\\
a_{13} &= \sin\psi\sin\vartheta,\\
a_{21} &= \sin\psi\cos\varphi + \cos\psi\cos\vartheta\sin\varphi,\\
a_{22} &= -\sin\psi\sin\varphi + \cos\psi\cos\vartheta\cos\varphi,\\
a_{23} &= -\cos\psi\sin\vartheta,\\
a_{31} &= \sin\vartheta\sin\varphi,\\
a_{32} &= \sin\vartheta\cos\varphi,\\
a_{33} &= \cos\vartheta.
\end{aligned} \tag{1.1}$$

Then, the equations of motion of the gyrostat satellite relative to the center of mass can be written as [4, 11]

$$\begin{aligned}
&A\dot{p} + (C-B)qr - 3\omega_0^2(C-B)a_{32}a_{33} - \bar{H}_2 r + \bar{H}_3 q = 0,\\
&B\dot{q} + (A-C)rp - 3\omega_0^2(A-C)a_{33}a_{31} - \bar{H}_3 p + \bar{H}_1 r = 0,\\
&C\dot{r} + (B-A)pq - 3\omega_0^2(B-A)a_{31}a_{32} - \bar{H}_1 q + \bar{H}_2 p = 0;
\end{aligned} \tag{1.2}$$

$$\begin{aligned}
p &= \bar{p} + \omega_0 a_{21}, \quad & \bar{p} &= \dot{\psi}a_{31} + \dot{\vartheta}\cos\varphi,\\
q &= \bar{q} + \omega_0 a_{22}, \quad & \bar{q} &= \dot{\psi}a_{32} - \dot{\vartheta}\sin\varphi,\\
r &= \bar{r} + \omega_0 a_{23}, \quad & \bar{r} &= \dot{\psi}a_{33} + \dot{\varphi}.
\end{aligned} \tag{1.3}$$

In Eqs. (1.2)–(1.3), A, B, C are the principal central moments of inertia of the gyrostat; p, q, r and $\bar{H}_1, \bar{H}_2, \bar{H}_3$ are the projections of absolute angular velocity of the gyrostat and constant projections of the vector gyrostatic moment on the axes Ox, Oy, Oz, respectively; ω_0 is the angular velocity of the motion of the gyrostat's center of mass in a circular orbit. The dot denotes differentiation with respect to time t.

For equations of motion (1.2) and (1.3), the following generalized energy integral holds [4]:

$$\begin{aligned}&\frac{1}{2}(A\bar{p}^2 + B\bar{q}^2 + C\bar{r}^2) + \frac{3}{2}\omega_0^2[(A - C)a_{31}^2 + (B - C)a_{32}^2] \\ &+ \frac{1}{2}\omega_0^2[(B - A)a_{21}^2 + (B - C)a_{23}^2] - \omega_0(\bar{H}_1 a_{21} + \bar{H}_2 a_{22} + \bar{H}_3 a_{23}) = \text{const}.\end{aligned} \tag{1.4}$$

2. EQUILIBRIUM ATTITUDES OF THE GYROSTAT SATELLITE

Assuming in (1.2) and (1.3) that $\psi = \psi_0 = \text{const}$, $\vartheta = \vartheta_0 = \text{const}$, $\varphi = \varphi_0 = \text{const}$, and $H_i = \bar{H}_i/\omega_0$, $i = 1, 2, 3$, we obtain for $A \neq B \neq C$ the equations

$$\begin{aligned}&(C - B)(a_{22}a_{23} - 3a_{32}a_{33}) - H_2 a_{23} + H_3 a_{22} = 0, \\ &(A - C)(a_{23}a_{21} - 3a_{33}a_{31}) - H_3 a_{21} + H_1 a_{23} = 0, \\ &(B - A)(a_{21}a_{22} - 3a_{31}a_{32}) - H_1 a_{22} + H_2 a_{21} = 0,\end{aligned} \tag{2.1}$$

which make it possible to determine the equilibrium attitudes of the gyrostat satellite in the orbital coordinate system. Hereinafter, it is more convenient to use the equivalent system

$$\begin{aligned}&4(Aa_{21}a_{31} + Ba_{22}a_{32} + Ca_{23}a_{33}) + (H_1 a_{31} + H_2 a_{32} + H_3 a_{33}) = 0, \\ &Aa_{11}a_{31} + Ba_{12}a_{32} + Ca_{13}a_{33} = 0, \\ &(Aa_{11}a_{21} + Ba_{12}a_{22} + Ca_{13}a_{23}) + (H_1 a_{11} + H_2 a_{12} + H_3 a_{13}) = 0,\end{aligned} \tag{2.2}$$

which is obtained by projecting Eqs. (2.1) on the axes of the orbital coordinate system. Using the dimensionless parameters $h_i = H_i/(B - C)$ $(i = 1, 2, 3)$, and $\nu = (B - A)/(B - C)$, system (2.2) can be represented as

$$\begin{aligned}&-4(\nu a_{21}a_{31} + a_{23}a_{33}) + (h_1 a_{31} + h_2 a_{32} + h_3 a_{33}) = 0, \\ &\nu a_{11}a_{31} + a_{13}a_{33} = 0, \\ &\nu a_{11}a_{21} + a_{13}a_{23} - (h_1 a_{11} + h_2 a_{12} + h_3 a_{13}) = 0.\end{aligned} \tag{2.3}$$

In view of (1.1), system (2.2) or (2.3) can be regarded as a system of three equations with unknown variables ψ_0, ϑ_0, φ_0. Another method for closing Eqs. (2.2) is to add six conditions of orthogonality for direction cosines (1.1):

$$\begin{aligned}&a_{11}^2 + a_{12}^2 + a_{13}^2 = 1, \quad a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0, \\ &a_{21}^2 + a_{22}^2 + a_{23}^2 = 1, \quad a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0, \\ &a_{31}^2 + a_{32}^2 + a_{33}^2 = 1, \quad a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0.\end{aligned} \tag{2.4}$$

Equations (2.2) and (2.4) were solved for some special cases. For the case when the vector of the gyrostatic moment is parallel to one of the principal central axes of inertia of the gyrostat satellite (for example, $h_2 \neq 0$, $h_1 = h_3 = 0$, $A \neq B \neq C$), all equilibrium attitudes were analytically determined depending on two dimensionless parameters of the problem and sufficient conditions for the stability of these equilibrium attitudes were obtained as simple inequalities [6–8]. The solution of the problem for the case when the vector of the gyrostatic moment is parallel to the plane of any two principal central axes of inertia (for example, $h_1 \neq 0$, $h_2 = 0$, $h_3 \neq 0$, $A \neq B \neq C$) was considered in [5, 9]. Finally, the equilibrium attitudes and their stability for the case of an axisymmetric gyrostat satellite ($h_1 \neq 0$, $h_2 \neq 0$, $h_3 \neq 0$, $A \neq B = C$) were analyzed in [10].

Then, we consider the equilibrium attitudes of the gyrostat satellite in the general case when $h_1 \neq 0$, $h_2 \neq 0$, $h_3 \neq 0$, $A \neq B \neq C$. Equations (2.2) and (2.4) form a closed algebraic system of equations for nine direction cosines controlling the equilibrium attitudes of the gyrostat satellite. For this system of equations, we formulate the following (direct) problem: for given values of A, B, C, H_1, H_2, H_3, it is required to determine all the nine direction cosines, i.e., all the equilibrium attitudes of the gyrostat satellite in the orbital coordinate system.

As shown in [11], Eqs. (2.2) and (2.4) can be solved with respect to $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$ for $A \neq B \neq C$ in the following way:

$$\begin{aligned} &a_{11} = -4a_{32}a_{33}/F, \quad a_{21} = 4[\nu a_{32}^2 - (1-\nu)a_{33}^2]a_{31}/F,\\ &a_{12} = 4(1-\nu)a_{33}a_{31}/F, \quad a_{22} = -4(\nu a_{31}^2 + a_{33}^2)a_{32}/F,\\ &a_{13} = 4\nu a_{31}a_{32}/F, \quad a_{23} = 4[(1-\nu)a_{31}^2 + a_{32}^2]a_{33}/F, \end{aligned} \tag{2.5}$$

where $F = h_1a_{31} + h_2a_{32} + h_3a_{33}$.

Substituting Eq. (2.5) into the first and third equations of (2.3) and adding the third equation of (2.4), we obtain the system of equations [11, 12]

$$\begin{aligned} &16[a_{32}^2a_{33}^2 + (1-\nu)^2a_{33}^2a_{31}^2 + \nu^2a_{31}^2a_{32}^2] = (h_1a_{31} + h_2a_{32} + h_3a_{33})^2(a_{31}^2 + a_{32}^2 + a_{33}^2),\\ &4\nu(1-\nu)a_{31}a_{32}a_{33} + [h_1a_{32}a_{33} - h_2(1-\nu)a_{33}a_{31} - h_3\nu a_{31}a_{32}](h_1a_{31} + h_2a_{32} + h_3a_{33}) = 0,\\ &a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \end{aligned} \tag{2.6}$$

for determining the direction cosines a_{31}, a_{32}, a_{33}. On solving system (2.6), formulas (2.5) make it possible to determine the remaining six direction cosines. It should be noted that the right-hand side of the first equation in (2.6) was multiplied by $a_{31}^2 + a_{32}^2 + a_{33}^2 = 1$ to ensure homogeneity.

In view of the homogeneity of the first two equations in (2.6), we divide both sides of the first equation by a_{33}^4 and those of the second equation by a_{33}^3 to obtain an algebraic system of two equations with respect to the variables $x = a_{31}/a_{33}$, $y = a_{32}/a_{33}$:

$$\begin{aligned} &16[y^2 + (1-\nu)^2x^2 + \nu^2x^2y^2] = (h_1x + h_2y + h_3)^2(1 + x^2 + y^2),\\ &4\nu(1-\nu)xy + [h_1y - h_2(1-\nu)x - h_3\nu xy](h_1x + h_2y + h_3) = 0. \end{aligned} \tag{2.7}$$

Then, substituting the expressions $a_{31} = xa_{33}$, $a_{32} = ya_{33}$ into the last equation of system (2.6), we obtain the expression

$$a_{33}^2 = \frac{1}{1 + x^2 + y^2}. \tag{2.8}$$

Equations (2.7) can be represented as

$$\begin{aligned} &a_0y^2 + a_1y + a_2 = 0,\\ &b_0y^4 + b_1y^3 + b_2y^2 + b_3y + b_4 = 0, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} &a_0 = h_2(h_1 - \nu h_3x),\\ &a_1 = h_1h_3 + [4\nu(1-\nu) + h_1^2 - (1-\nu)h_2^2 - \nu h_3^2]x - \nu h_1h_3x^2,\\ &a_2 = -(1-\nu)h_2(h_1x + h_3)x,\\ &b_0 = h_2^2, \quad b_1 = 2h_2(h_1x + h_3),\\ &b_2 = (h_2^2 + h_3^2 - 16) + 2h_1h_3x + (h_1^2 + h_2^2 - 16\nu^2)x^2,\\ &b_3 = 2h_2(h_1x + h_3)(1 + x^2), \quad b_4 = (h_1x + h_3)^2(1 + x^2) - 16(1-\nu)^2x^2. \end{aligned} \tag{2.10}$$

Excluding y from the system of two equations (2.9) with the help of the resultant concept, we obtain a 12th order algebraic equation with respect to x [12]:

$$\begin{aligned} &p_0x^{12} + p_1x^{11} + p_2x^{10} + p_3x^9 + p_4x^8 + p_5x^7 + p_6x^6\\ &+p_7x^5 + p_8x^4 + p_9x^3 + p_{10}x^2 + p_{11}x + p_{12} = 0, \end{aligned} \tag{2.11}$$

the coefficients of which are sufficiently complex polynomials depending on the parameters ν, h_1, h_2, h_3.

It should be noted that the number of real roots of algebraic equation (2.11) is even and does not exceed 12. Substituting the value of the real root x_1 of Eq. (2.11) into the equations of system (2.9), we find the coincident root y_1 of these equations. For each solution (x_1, y_1), one can find two values of a_{33} from Eq. (2.8)

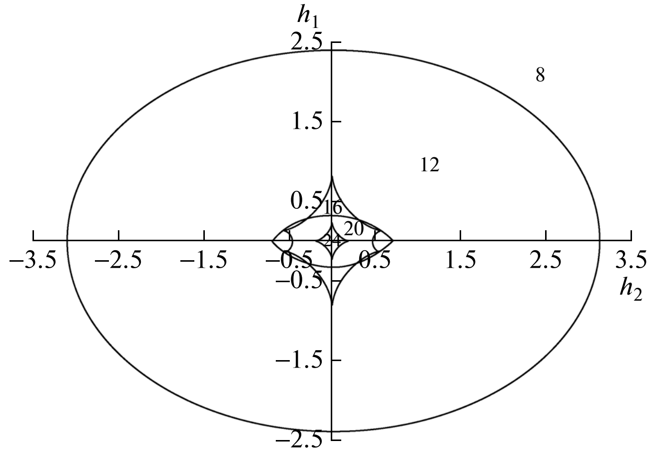

Fig. 1. $\nu = 0.2$, $h_3 = 0.25$.

and then their corresponding values $a_{31} = x_1 a_{33}$ and $a_{32} = y_1 a_{33}$. Thus, each real root of algebraic equation (2.11) corresponds to two sets of values a_{31}, a_{32}, a_{33}, which uniquely determine the remaining direction cosines $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$. in view of (2.5). It follows from here that in the general case ($h_1 \neq 0,\ h_2 \neq 0,\ h_3 \neq 0, A \neq B \neq C$) the satellite in a circular orbit under the action of gravitational moment can have no more than 24 equilibrium attitudes.

3. STUDY OF EQUILIBRIUM ATTITUDES OF THE GYROSTAT SATELLITE

Equations (2.5), (2.9), and (2.11) make it possible to determine all equilibrium attitudes of the gyrostat satellite subjected to the action of the gravitational moment for given values of the dimensionless parameters $\nu,\ h_1,\ h_2,\ h_3$.

In studying the equilibrium attitudes of the gyrostat satellite, one should determine (in the space of parameters) the domains with a same number of real roots of Eq. (2.11). The partition of the space of parameters into domains with a same number of real roots of the equation is determined by the discriminant hypersurface, which is given by the discriminant of polynomial (2.11). The symbolic study of the system of algebraic equations determining the set of singular points of the discriminant hypersurface seems to be impossible because the expressions for the coefficients of polynomial (2.11) are cumbersome.

The dependence of the number of real solutions of Eq. (2.11) on the parameters was studied in [12], which involves a detailed numerical analysis of the evolution of the domains of existence of different numbers of equilibrium attitudes in the space of dimensionless parameters. The numerical studies in [12] were performed for the conditions $B > A > C$ $(0 < \nu < 1)$. As an example, Fig. 1 shows domains with 24, 20, 16, 12, and 8 equilibrium attitudes for the parameter values $\nu = 0.2$ and $h_3 = 0.25$.

Using the results of [10], one can show that for the limiting cases $\nu = 0$ and $\nu = 1$ (the cases of an axisymmetric gyrostat satellite) the boundaries between the domains with a constant number of equilibrium attitudes are determined analytically.

For the axisymmetric case of the gyrostat satellite $\nu = 0$ $(A = B)$, the system of equations (2.3) is simplified and, as a result, one can obtain equations of two circles in the plane $(h_1,\ h_2)$:

$$
\begin{aligned}
&h_1^2 + h_2^2 = (4^{2/3} - h_3^{2/3})^3, \\
&h_1^2 + h_2^2 = (1 - h_3^{2/3})^3,
\end{aligned}
\tag{3.1}
$$

which identify the boundaries of domains with a constant number of equilibrium attitudes of the satellite (Fig. 2).

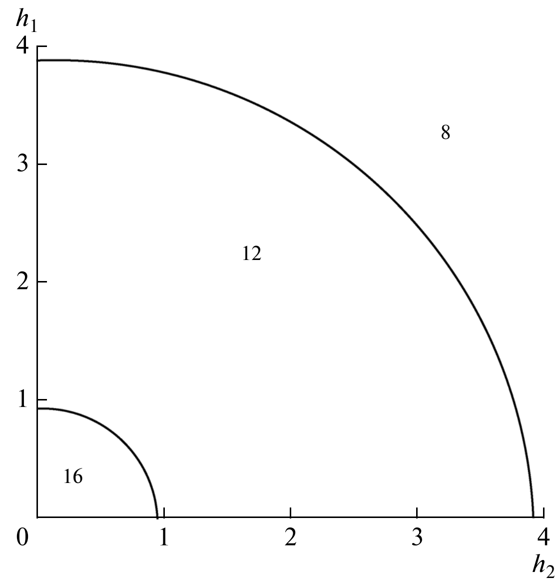

Fig. 2. $\nu = 0$, $h_3 = 0.01$ (axisymmetric case $A = B$).

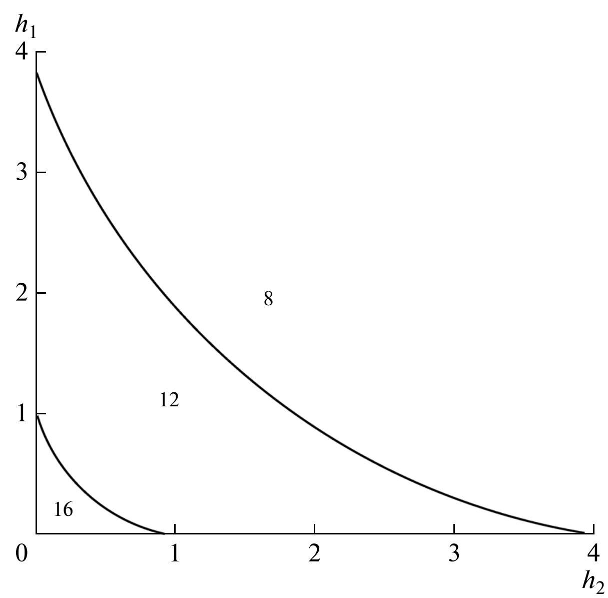

Fig. 3. $\nu = 1.0$, $h_3 = 0.01$ (axisymmetric case $A = C$).

For the axisymmetric case $\nu = 1$ $(A = C)$, the system of equations (2.3) is also simplified and then one can analytically obtain equations for two astroids in the plane $(h_1,\ h_2)$:

$$\begin{aligned} &h_2^{2/3} + (h_1^2 + h_3^2)^{1/3} = 4^{2/3}, \\ &h_2^{2/3} + (h_1^2 + h_3^2)^{1/3} = 1, \end{aligned} \tag{3.2}$$

identifying the domains with a constant number of equilibrium attitudes (Fig. 3).

As shown in [10], in the case of an axisymmetric gyrostat satellite, there exist only 16, 12, and 8 equilibrium attitudes, which correspond to three domains with the same number of equilibrium attitudes in the space of parameters.

4. STUDY OF SUFFICIENT CONDITIONS FOR THE STABILITY OF EQUILIBRIUM ATTITUDES OF THE GYROSTAT SATELLITE

Equations (2.11) and (2.9) coupled with system (2.5) make it possible to determine all equilibrium attitudes of the gyrostat satellite for the given values of the inertial parameter and the components of the gyrostatic moment vector.

To study the sufficient conditions for stability of the resulting equilibrium attitudes of system (2.3) and (2.4), we use generalized energy integral (1.4) as a Lyapunov function. This integral can be rewritten as

$$\begin{aligned}&\frac{1}{2}(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2) + \frac{1}{2}(B - C)\omega_0^2\{3[(1 - \nu)a_{31}^2 + a_{32}^2]\\&+ (\nu a_{21}^2 + a_{23}^2) - 2(h_1a_{21} + h_2a_{22} + h_3a_{23})\} = \text{const}.\end{aligned} \tag{4.1}$$

Let us present ψ, ϑ, φ as

$$\psi = \psi_0 + \overline{\psi},\ \vartheta = \vartheta_0 + \overline{\vartheta},\ \varphi = \varphi_0 + \overline{\varphi},$$

where $\overline{\psi}, \overline{\vartheta}, \overline{\varphi}$ are small deviations from the equilibrium attitude of the satellite $\psi = \psi_0 = \text{const}$, $\vartheta = \vartheta_0 = \text{const}$, and $\varphi = \varphi_0 = \text{const}$, respectively, satisfying system of equations (2.3). Then, energy integral (4.1) can be written as

$$\begin{aligned}&\frac{1}{2}(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2) + \frac{1}{2}(B - C)\omega_0^2(A_{\psi\psi}\overline{\psi}^2 + A_{\vartheta\vartheta}\overline{\vartheta}^2 + A_{\varphi\varphi}\overline{\varphi}^2\\&+ 2A_{\psi\vartheta}\overline{\psi}\overline{\vartheta} + 2A_{\psi\varphi}\overline{\psi}\overline{\varphi} + 2A_{\vartheta\varphi}\overline{\vartheta}\overline{\varphi}) + \Sigma = \text{const},\end{aligned} \tag{4.2}$$

where the symbol Σ denotes the terms with an order of smallness exceeding 2 with respect to $\overline{\psi}$, $\overline{\vartheta}$, $\overline{\varphi}$, and

$$\begin{aligned}&A_{\psi\psi} = \nu(a_{11}^2 - a_{21}^2) + (a_{13}^2 - a_{23}^2) + h_1a_{21} + h_2a_{22} + h_3a_{23},\\&A_{\vartheta\vartheta} = (3 + \cos^2\psi_0)(1 - \nu\sin^2\varphi_0)\cos 2\vartheta_0\\&-\frac{1}{4}\nu\sin 2\psi_0\cos\vartheta_0\sin 2\varphi_0 + (h_1\sin\varphi_0 + h_2\cos\varphi_0)\cos\psi_0\cos\vartheta_0 + h_3a_{23},\\&A_{\varphi\varphi} = \nu[(a_{22}^2 - a_{21}^2) - 3(a_{32}^2 - a_{31}^2)] + h_1a_{21} + h_2a_{22},\\&A_{\psi\vartheta} = -\frac{1}{2}\sin 2\psi_0\sin 2\vartheta_0 + \nu(a_{11}a_{23} + a_{13}a_{21}) - \sin\psi_0(h_1a_{31} + h_2a_{32} + h_3a_{33}),\\&A_{\psi\varphi} = \nu(a_{11}a_{22} + a_{12}a_{21}) - h_1a_{12} + h_2a_{11},\\&A_{\vartheta\varphi} = -\frac{3}{2}\nu\sin 2\vartheta_0\sin 2\varphi_0 + \nu(a_{21}\cos\varphi_0 + a_{22})a_{23} - (h_1\cos\varphi_0 - h_2\sin\varphi_0)a_{23}.\end{aligned} \tag{4.3}$$

It follows from Lyapunov's theorem that the equilibrium attitude is stable if the quadratic form

$$\begin{aligned}&\frac{1}{2}(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2) + \frac{1}{2}(B - C)\omega_0^2(A_{\psi\psi}\overline{\psi}^2 + A_{\vartheta\vartheta}\overline{\vartheta}^2 + A_{\varphi\varphi}\overline{\varphi}^2\\&+ 2A_{\psi\vartheta}\overline{\psi}\overline{\vartheta} + 2A_{\psi\varphi}\overline{\psi}\overline{\varphi} + 2A_{\vartheta\varphi}\overline{\vartheta}\overline{\varphi})\end{aligned} \tag{4.4}$$

is positive definite. We write the sufficient conditions for stability as inequalities

$$\begin{aligned}&A_{\psi\psi} > 0,\\&A_{\psi\psi}A_{\vartheta\vartheta} - (A_{\psi\vartheta})^2 > 0,\\&A_{\psi\psi}A_{\vartheta\vartheta}A_{\varphi\varphi} + 2A_{\psi\vartheta}A_{\psi\varphi}A_{\vartheta\varphi} - A_{\psi\psi}(A_{\vartheta\varphi})^2 - A_{\vartheta\vartheta}(A_{\psi\varphi})^2 - A_{\varphi\varphi}(A_{\psi\vartheta})^2 > 0.\end{aligned} \tag{4.5}$$

Substituting the expressions $A_{\psi\psi}$, $A_{\vartheta\vartheta}$, $A_{\varphi\varphi}$, $A_{\psi\vartheta}$, $A_{\psi\varphi}$, $A_{\vartheta\varphi}$ of (4.3) for the corresponding equilibrium attitude in (4.5), we obtain sufficient conditions for the stability of this solution.

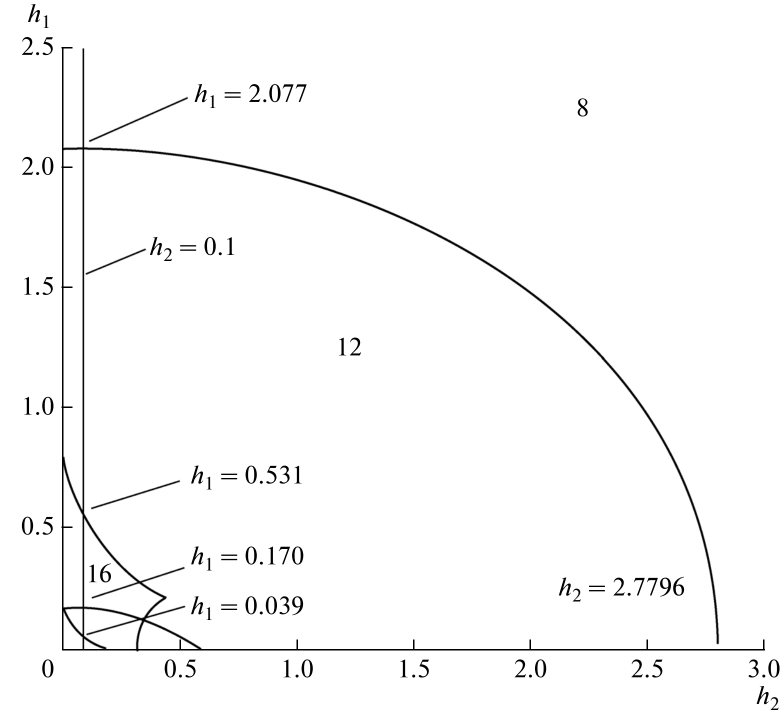

Fig. 4. $\nu = 0.2$, $h_3 = 0.4$.

For each set of parameters ν, h_1, h_2, h_3, we used the *Mathematica* numerical package to find the real root of Eq. (2.11). For clarity of the representation of equilibrium attitudes of the gyrostat satellite, we express the calculation results through Euler angles. Then, we have

$$\mathrm{tg}\varphi = \frac{a_{31}}{a_{32}} = \frac{x_1}{y_1} \quad \text{and} \quad \varphi = \mathrm{arctg}\frac{a_{31}}{a_{32}} \ (0 \le \varphi < 2\pi).$$

The angle ϑ $(0 \le \vartheta < \pi)$ is determined by the relation $\cos\vartheta = a_{33}$. The angle ψ $(0 \le \psi < 2\pi)$ is also uniquely determined from relations (1.1):

$$\sin\psi = \frac{a_{13}}{\sin\vartheta}; \ \cos\psi = -\frac{a_{23}}{\sin\vartheta}.$$

Thus, for each real root of (2.11), one can uniquely find two sets of orientation angles ψ_0, ϑ_0, φ_0, calculate the coefficients of quadratic form (4.3), and check the conditions of its positive definiteness (4.5).

Since $0 \le \varphi < 2\pi$, for each real root $\mathrm{tg}\varphi = x_1/y_1$, there exist two values φ (φ_1 and $\varphi_2 = \varphi_1 + \pi$). It follows from the properties of the coefficients of quadratic form (4.3) that the sufficient conditions for stability (4.5) for φ_1 and φ_2 are the same. In addition, it can be shown that conditions (4.5) are independent of the sign of parameters h_1, h_2, h_3. Consequently, the sufficient conditions for the stability of equilibrium solutions of Eqs. (2.3) can be numerically analyzed only for positive values of h_1, h_2, h_3, one value of angle φ (φ_1 or φ_2), and if the conditions $0 < \nu < 1$ are satisfied (Fig. 4).

Figures 5–14 show the calculated dependence of the angle φ on h_1 for fixed values of ν, h_2, and h_3; the dotted line denotes the branches of equilibrium attitudes for which sufficient stability conditions (4.5) are satisfied. Since the sufficient stability condition (4.5) for the angles φ_1 and $\varphi_1 + \pi$ $(0 \le \varphi < 2\pi)$ are the same, the numerical results in Figs. 5–14 are shown for $0 \le \varphi < \pi$.

The calculations were performed for the following values of the inertial parameter: $\nu = 0.2$, $\nu = 0.6$, $\nu = 0.7$, and $\nu = 0.9$ (Figs. 5–14). It follows from the analysis of the calculation results for the given values

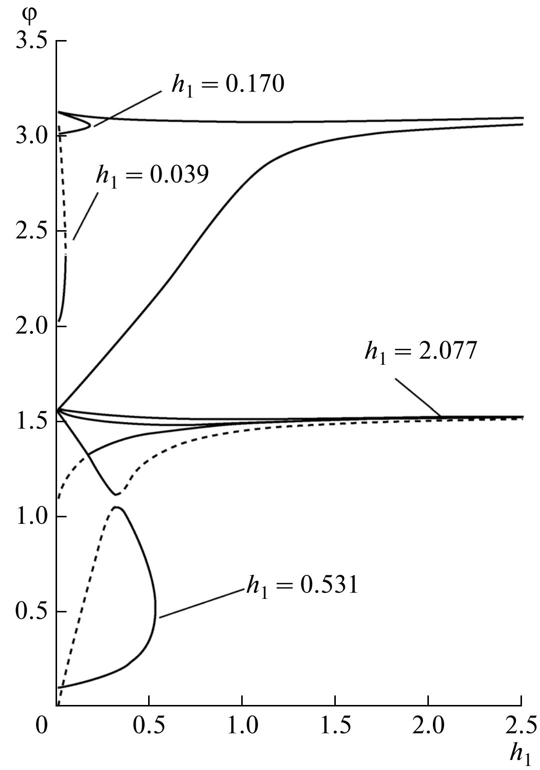

Fig. 5. $\nu = 0.2$, $h_2 = 0.1$, $h_3 = 0.4$ (24 equilibrium orientations, 4 stable orientations).

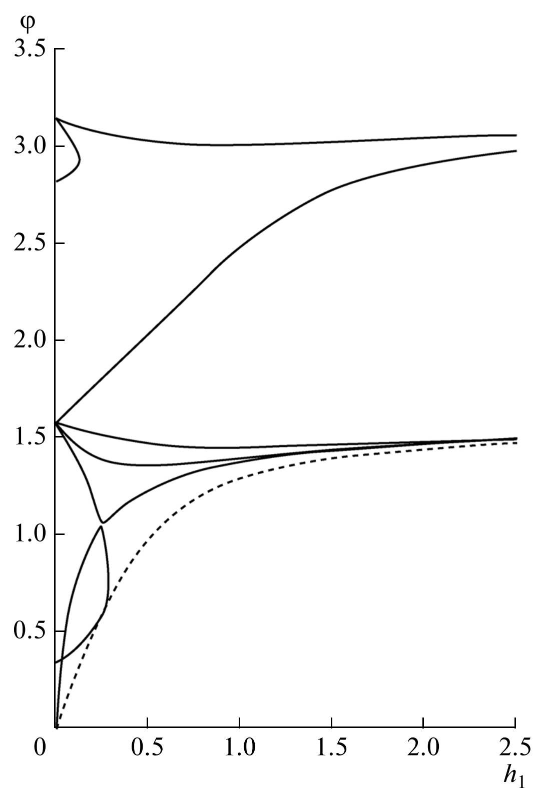

Fig. 6. $\nu = 0.2$, $h_2 = 0.3$, $h_3 = 0.4$ (20 equilibrium orientations, 2 stable orientations).

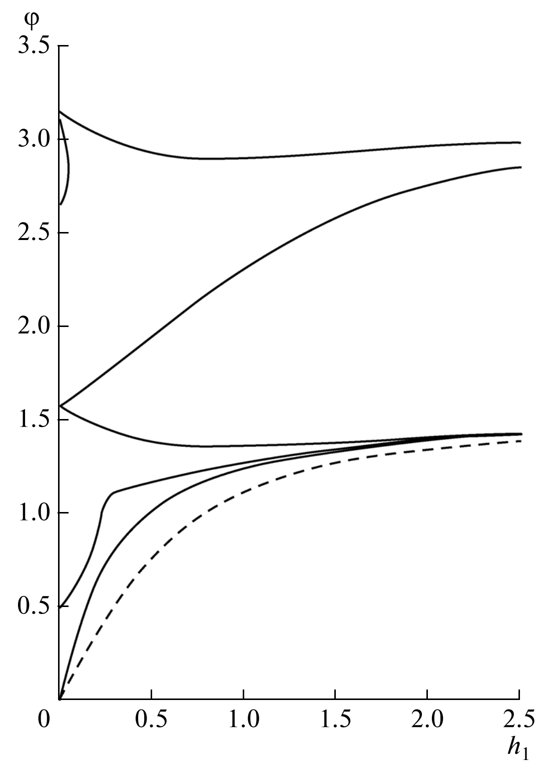

Fig. 7. $\nu = 0.2$, $h_2 = 0.5$, $h_3 = 0.4$ (16 equilibrium orientations, 2 stable orientations).

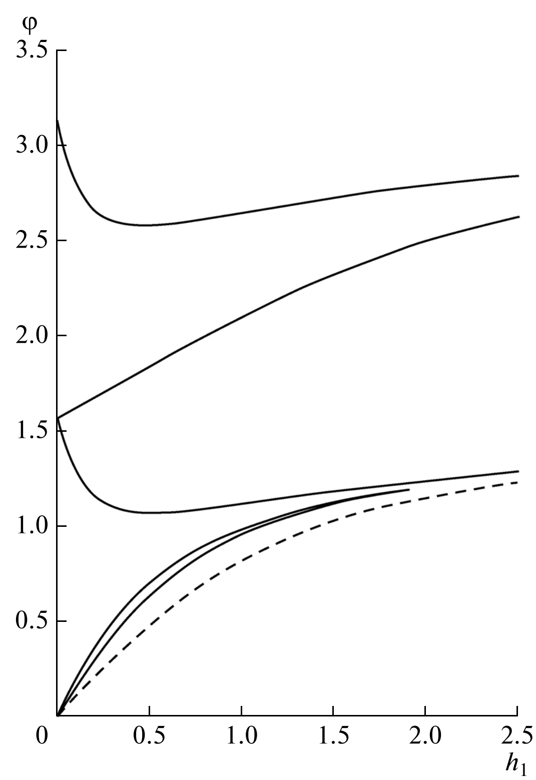

Fig. 8. $\nu = 0.2$, $h_2 = 1.0$, $h_3 = 0.4$ (12 equilibrium orientations, 2 stable orientations).

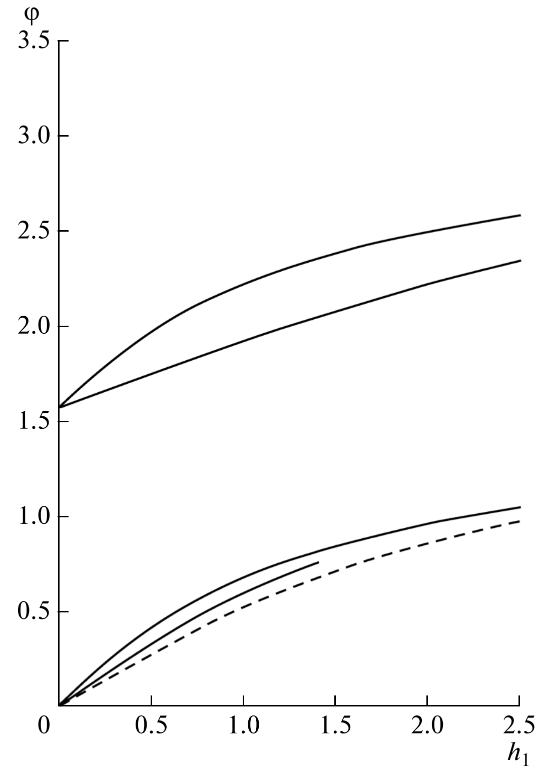

Fig. 9. $v = 0.2$, $h_2 = 2.0$, $h_3 = 0.4$ (12 equilibrium orientations, 2 stable orientations).

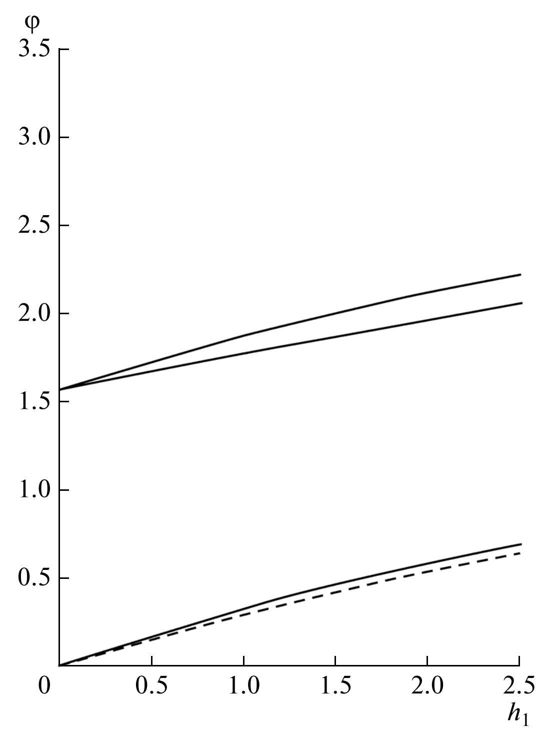

Fig. 10. $v = 0.2$, $h_2 = 4.0$, $h_3 = 0.4$ (8 equilibrium orientations, 2 stable orientations).

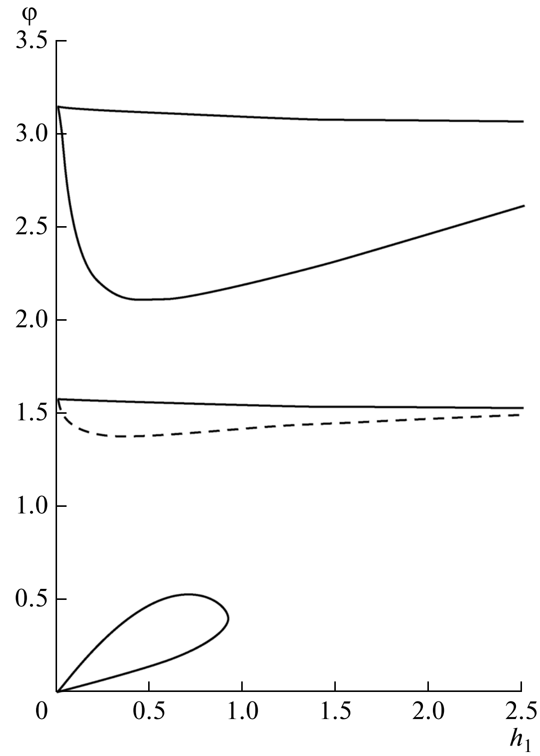

Fig. 11. $v = 0.6$, $h_2 = 0.5$, $h_3 = 2.0$ (12 equilibrium orientations, 4 stable orientations).

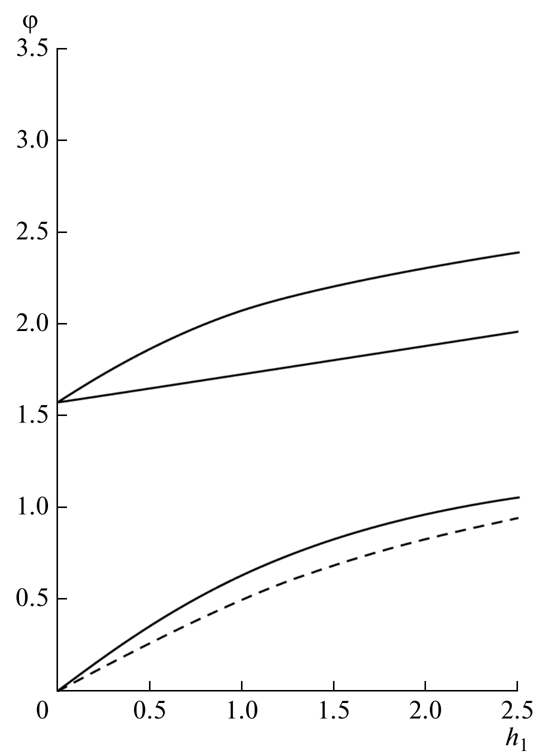

Fig. 12. $v = 0.6$, $h_2 = 4.0$, $h_3 = 0.01$ (8 equilibrium orientations, 2 stable orientations).

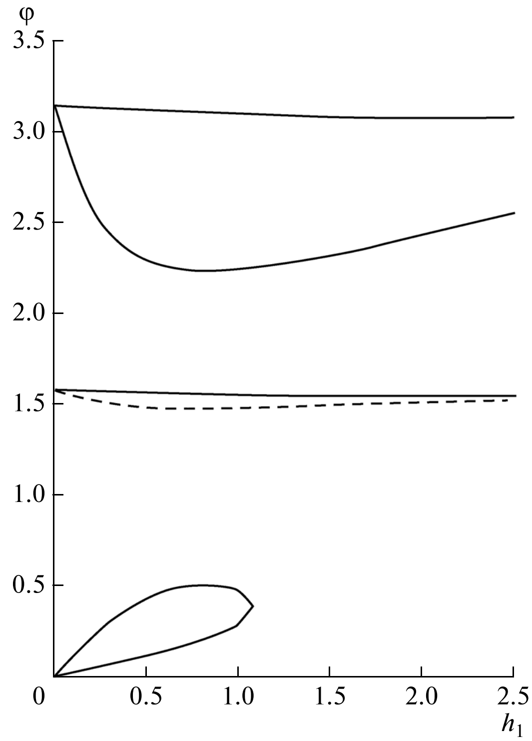

Fig. 13. $\nu = 0.7$, $h_2 = 0.5$, $h_3 = 2.0$ (12 equilibrium orientations, 4 stable orientations).

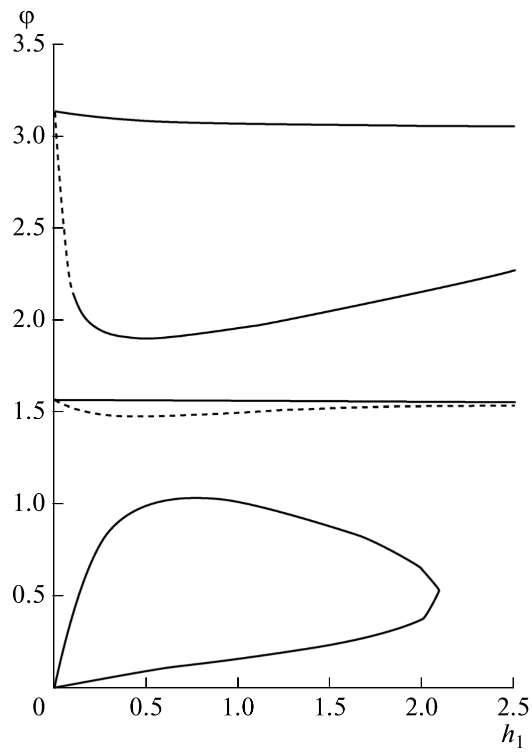

Fig. 14. $\nu = 0.9$, $h_2 = 0.5$, $h_3 = 1.0$ (12 equilibrium orientations, 4 stable orientations).

of the parameters that for $h_3 < 1 - \nu$ and for small values of h_1, h_2 there exist 24 branches of equilibrium attitudes, and the stability conditions (4.5) are satisfied for four branches of these 24 (Fig. 5). There exist also four stable equilibrium attitudes for $\nu > 0.5$ and $h_3 \geq 1 - \nu$ (Figs. 11, 13, and 14).

When the parameter h_1 increases, the branches of equilibrium attitudes gradually merge at the points corresponding to the points of intersection of line h_2 = const with the boundaries of the domains with an equal number of equilibrium attitudes. For example, Fig. 4 ($\nu = 0.2$, $h_3 = 0.4$) shows that there exist four points of intersection of the line $h_2 = 0.1$ with the boundaries of an equal number of equilibrium attitudes $h_1 = 0.039$, $h_1 = 0.17$, $h_1 = 0.531$, and $h_1 = 2.077$; Fig. 5 ($\nu = 0.2$, $h_2 = 0.1$, $h_3 = 0.4$) shows that at these points the branches of the equilibrium attitudes merge.

For the values of gyrostatic moment parameters h_1, h_2, h_3 greater or equal to four, there exist only eight equilibrium attitudes (Figs. 10 and 12) and only two of them are stable. For large values of h_1, h_2, h_3, the equilibrium values of the angle φ approach trivial solutions, when one of the axes of the orbital coordinate system coincides with one of the axes of the coordinate system bound with the satellite. The character of stability of equilibrium attitudes corresponds (depending on the problem parameters) to the character of the stability of equilibrium attitudes for the axisymmetric case [10]; here, it was shown that the number of equilibrium attitudes of the gyrostat satellite for which the sufficient conditions for stability are satisfied varies from four to two as the magnitude of the gyrostatic moment increases (as in the general case).

CONCLUSIONS

This is a study of the rotational motion of a gyrostat satellite relative to the center of mass in a circular orbit under the action of gravitational moment. The emphasis is on the study of the stability of equilibrium attitudes of the gyrostat satellite. A symbolic-numerical method is proposed for determining all the equilibrium attitudes of the gyrostat satellite in the orbital coordinate system for the given values of the gyrostatic moment vector and principal central moments of inertia in the general case, when $A \neq B \neq C$ and $h_1 \neq 0$, $h_2 \neq 0$, $h_3 \neq 0$. It is shown that the number of equilibrium attitudes of the gyrostat satellite in a circular orbit in the general case is less than or equal to 24 and cannot be less than eight.

For each equilibrium attitude, sufficient conditions for stability are applied using a generalized energy integral such as a Lyapunov function. A detailed numerical analysis of the domains that hold the conditions of stability of equilibrium attitudes is conducted depending on four dimensionless parameters of the problem. It is shown that the number of equilibrium attitudes of the gyrostat satellite for which the sufficient conditions for stability are satisfied varies from four to two (in the general case) as the gyrostatic moment magnitude increases. The results obtained in this paper can be used for the preliminary construction of passive gravitational systems of control over the orientation of a satellite.

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Translated by V. Arutyunyan