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Potential 2D Flows of Real Gases. Invariant Equations in Natural Coordinates. Hodograph Transformation

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Abstract
We consider plane-parallel and axially symmetric stationary potential flows of real gases. We deduce the motion equations in the natural coordinates that are invariant with respect to thermodynamic state laws. This allows to extend, almost with no change, methods of classical gas dynamics to a broad class of urgent problems. We generalize the Chaplygin transformation onto flows of real gases (combustion gases, superheated steam etc.). In particular, this will allow to apply our method of aerodynamic designing for precise constructing gas and steam turbines as well as jet nozzles.
The obtained equation in the Chaplygin variables differs from the classical one only by the coefficients that depend on concrete thermodynamic laws. The equation of axially symmetric flows has in addition right-side part that contains the transformation's Jacobian.

Keywords: Gas dynamics, aerodynamic designing, real gas, hodograph, 2D-flows, first integral, stationary, potential, stream function, separated-free, Chaplygin's equation, well-posed problem.

Introduction

Plane-parallel and axially symmetrical potential flows of an real gas take place in diverse technical devices. In this work, we specify and expand results of our work [1], bearing in mind mainly the problem of aerodynamic designing. As the most important employments we mean flows of the superheated steam and products of fuel combustion in turbines, as well as in nozzles of rocket engines. In these devices dissociation takes place because of high temperature, therefore a quality designing cannot be attained without taking into account real nature of gases.

When designing gas dynamic device, they are searching such form of a body that provides a steady, separation-free flowing. As a rule, this problem is being solved by selection of suitable solutions of the direct problem defined on a set of close bodies. (This problem consisting in determination of the flow around a given body is assumed to be uniquely solvable.) However the hard condition for the flow to be separation-free requires rather thin adjustment of search strategy. Therefore this approach is not always successful.

In the book [2] we have developed another method of aerodynamic designing based on the hodograph-transformation of potential 2D-flows of a perfect (Clapeyron's) gas. This transformation has turned out extremely useful since transition to the velocity variables has made the designing problem well-posed. Firstly, hodograph-image of the flow domain can be set enough arbitrarily. This makes possible to satisfy in advance the condition for the velocity distribution along a body to be increasing (or not too strongly decreasing). By the boundary layer theory [3] this provides a separation-free flow at any Re number and consequently guarantees adequacy of the mathematical model of ideal gas. Secondly, the designing problem can be formulated for the Chaplygin equation as well-posed one.

Initially this method has been employed for designing contours of the Laval nozzles of supersonic aerodynamic tubes [2]. Namely, the shortest nozzles with angular points at intersections of the contours with the rectilinear sonic lines have been constructed. These nozzles were manufactured and are in exploitation in Russia since 1980; the checking has shown that non-uniformity in the Mach number in these tubes does not exceed 1%.

After that the following devices have been designed:
a lattice of the turbine nozzle guide vanes [4]
an subcritical lifting wing section for flight with high subsonic velocity [5]
an inlet valve of a piston engine [6],[7]
a ring-shaped Laval nozzle [8].

Now, to expand the method scope, we deduce invariant equations of imperfect *non-barotropic* gases in the physical and hodograph variables. (Hodograph-mapping of plane-parallel flows of barotropic gases was described by R.Mises [9].)

First integrals of stationary 3D-flows of inviscid fluids

In this section we specify some results given in the monograph [9], Chapter V. (Adiabatic flow of inviscid fluid).

By definition, a first integral is an explicit expression satisfying a differential equation. Using first integrals, one can simplify mathematical problem.

From view point of the classical mechanics, homogeneous gas is a two-parametric locally-balanced thermodynamic medium: all thermodynamic parameters (the temperature T, pressure p, density ρ, interior energy E and so on) obey classic thermodynamics and can be expressed via any two of them [10]. We choose entropy S and enthalpy $I = E + p/\rho$ as independent variables. By Π denote the ratio p/ρ. We consider the differentiable functions $\Pi(I,S)$ and $T(I,S)$ as laws of the thermodynamic state. We write down the first law of thermodynamics as follows

$$d(\ln\rho(I,S))=\frac{dE(I,S)-T(I,S)dS}{\Pi(I,S)}=$$
$$=\frac{dI-d\Pi(I,S)-T(I,S)dS}{\Pi(I,S)}= \qquad (1)$$
$$=\frac{1-\Pi_I(I,S)}{\Pi(I,S)}dI-\frac{T(I,S)+\Pi_S(I,S)}{\Pi(I,S)}dS$$

The thermodynamic sound velocity a is determined by the formula

$$a^2=\left.\frac{dp(I,S)}{d\rho(I,S)}\right|_{S=const=S_0}=\Pi(I,S)\left.\frac{d\ln p(I,S)}{d\ln\rho(I,S)}\right|_{S=const=S_0}= \qquad (2)$$
$$=\Pi(I,S_0)\frac{\partial(\ln\rho(I,S_0))/\partial I+\partial(\ln\Pi(I,S_0))/\partial I}{\partial(\ln\rho(I,S_0))/\partial I}$$

It follows from eq.(1)

$$d(\ln\rho(I,S_0))=\frac{\partial(\ln\rho(I,S_0))}{\partial I}dI=\frac{1-\partial\Pi(I,S_0)/\partial I}{\Pi(I,S_0)}dI$$

Substituting this formula into eq. (2), we obtain

$$a^2=\frac{\Pi(I,S_0)}{1-\partial\Pi(I,S_0)/\partial I} \qquad (3)$$

Therefore eq.(1) takes the form

$$d\ln\rho(I,S)=\frac{dI}{a^2(I,S_0)}-\frac{T(I,S)+\Pi_S(I,S)}{\Pi(I,S)}dS \qquad (1')$$

A stationary 3D-flow of an ideal imperfect gas obeys the equations

$$\mathrm{div}(\rho\boldsymbol{V})=0,\ \rho(\boldsymbol{V}\cdot\nabla)\boldsymbol{V}+\nabla p=0,\ \rho\boldsymbol{V}\cdot\nabla\left(E+|\boldsymbol{V}|^2/2\right)+\mathrm{div}(p\boldsymbol{V})=0 \quad (4)$$

Here $\boldsymbol{V}=u\boldsymbol{i}+v\boldsymbol{j}$ is a flow velocity, $|\boldsymbol{V}|=V$. The functions $\rho=\rho(I,S),\ p=p(I,S),\ E=E(I,S)$ are defined by thermodynamics of an real (imperfect) gas.

The full temperature $T_0=T|_{V=0}$ and entropy in the Clapeiron gas are known to be constant on stream lines. The analogous property takes place in imperfect gas.

(By definition, the stream lines are vector lines of the velocity field $\boldsymbol{V}(x_1,x_2,x_3)$.)

Theorem 1. System (4) has two first integrals: entropy S and "full enthalpy" $I_0=I+V^2/2=I|_{V=0}$ are constants on stream lines, i.e. $\boldsymbol{V}\cdot\nabla I_0=0,\ \boldsymbol{V}\cdot\nabla S=0$.

Proof. First transform the energy equation using the continuity equation

$$\boldsymbol{V}\cdot\nabla\left(E+\frac{V^2}{2}\right)+\frac{1}{\rho}\mathrm{div}(p\boldsymbol{V})=0\ \Rightarrow \qquad (5)$$
$$\boldsymbol{V}\cdot\nabla\left(I+\frac{V^2}{2}\right)-\boldsymbol{V}\cdot\nabla\Pi+\frac{1}{\rho}\mathrm{div}(\rho\Pi\boldsymbol{V})=\boldsymbol{V}\cdot\nabla\left(I+\frac{V^2}{2}\right)=\boldsymbol{V}\cdot\nabla I_0=0$$

Now consider the momentum equation

$$\rho\nabla(|\boldsymbol{V}|^2/2)-\rho\boldsymbol{V}\times\mathrm{rot}\boldsymbol{V}+\nabla p=0$$

Multiplying on $\boldsymbol{V}$ and taking into account eq.(5), we obtain

$$\rho\boldsymbol{V}\cdot\nabla(|\boldsymbol{V}|^2/2)+\boldsymbol{V}\cdot\nabla p=0\ \Rightarrow$$
$$\boldsymbol{V}\cdot\nabla i=\rho^{-1}\boldsymbol{V}\cdot\nabla p=\boldsymbol{V}\cdot\nabla\Pi+\Pi\boldsymbol{V}\cdot\nabla\ln\rho\Rightarrow$$
$$\boldsymbol{V}\cdot\nabla E=\Pi\boldsymbol{V}\cdot\nabla\ln\rho$$

Comparing with eq.(1) written in the form

$$\boldsymbol{V}\cdot\nabla E=T\boldsymbol{V}\cdot\nabla S+\Pi\boldsymbol{V}\cdot\nabla\ln\rho$$

we obtain that $\boldsymbol{V}\cdot\nabla S=0$. Theorem 1 is proved.

Theorem 2. Let us suppose that: (**i**) vectors $\boldsymbol{V}$ and $rot\boldsymbol{V}$ are not collinear in Q and (**ii**) $\partial T(I,S)/\partial I\neq 0$ identically. Then $\boldsymbol{V}=\nabla\varphi$ if and only if $I_0=const,\ S=const$.

Proof. Let $\boldsymbol{n}$ be arbitrary unit vector, $\boldsymbol{n}\perp\boldsymbol{V}$. It follows from eq. (1') that

$$\boldsymbol{n}\cdot\nabla\ln\rho=\frac{1}{a^2(I,S)}\boldsymbol{n}\cdot\nabla(I_0-V^2/2)-\frac{T(I,S)+\Pi_S(I,S)}{\Pi(I,S)}\boldsymbol{n}\cdot\nabla S$$

Using formula (3), we obtain

$$\frac{1}{\rho}\boldsymbol{n}\cdot\nabla p=\boldsymbol{n}\cdot\nabla\Pi+\Pi\boldsymbol{n}\cdot\nabla\ln\rho=\boldsymbol{n}\cdot\nabla(I_0-V^2/2)-T\boldsymbol{n}\cdot\nabla S \qquad (6)$$

Taking scalar product of the momentum equation and the vector $\boldsymbol{n}$, we obtain

$$(\rho\boldsymbol{V}\times rot\boldsymbol{V})\cdot\boldsymbol{n}=\rho\boldsymbol{n}\cdot\nabla\left(V^2/2\right)+\boldsymbol{n}\cdot\nabla p$$

Taking into account the condition (**i**) and comparing with eq.(6), we have

$$rot\boldsymbol{V}=0\ \Leftrightarrow\ \boldsymbol{n}\cdot\nabla I_0-T\boldsymbol{n}\cdot\nabla S=0$$

By virtue of arbitrariness of vector $\boldsymbol{n}$ it follows from here that $\nabla I_0=0\ \Leftrightarrow\ \nabla S=0$. Therefore if $|\nabla I_0|^2+|\nabla S|^2>0$, then ∇I_0 and ∇S are collinear, consequently we obtain

$$\nabla I_0=T\nabla S\ \Leftrightarrow\ T=T(S)$$

This contradicts to the condition (**ii**). Theorem 2 is proved.

Invariant form of equations of 2D-potential flows

Transform system (4),(5) for plane-parallel (N=0) and axially symmetrical (N=1) potential flows. Let the potential $\varphi=\varphi(x,y),\nabla\varphi=\boldsymbol{V}=u\boldsymbol{i}+v\boldsymbol{j}$ and stream function $\psi=\psi(x,y),\ \nabla\psi=\rho y^N(-v\boldsymbol{i}+u\boldsymbol{j})$ form curvilinear orthogonal coordinate system (φ,ψ). Here $\boldsymbol{i},\boldsymbol{j}$ are unit vectors of Cartesian coordinate system (x,y) combined (in the axially symmetric flow) with the symmetry axis $y=0$. In correspondence with the Theorem 2 we have

$$I_0=I_0(\psi)=const=I_{00},\ S(\psi)=const=S_0$$

Let us denote $\boldsymbol{n}_1=\boldsymbol{V}/V$. By $\boldsymbol{n}_2$ denote $\boldsymbol{n}_1$ rotated on $\pi/2$ counter-clockwise. If $\boldsymbol{V}=Ve^{i\beta}$, then

$$\boldsymbol{n}_1=\boldsymbol{i}\cos\beta+\boldsymbol{j}\sin\beta,\ \boldsymbol{n}_2=-\boldsymbol{i}\sin\beta+\boldsymbol{j}\cos\beta$$

By $\partial/\partial s_{1,2}=\boldsymbol{n}_{1,2}\cdot\nabla$ denote directional derivatives. In the points, where $\rho\boldsymbol{V}\neq 0$ the first equation of system (4) can be transformed to the form

$$\mathrm{div}\boldsymbol{n}_1+\frac{\partial\ln(\rho V)}{\partial s_1}=0$$

Differential geometry says that

$$\mathrm{div}\boldsymbol{n}_1=\frac{\partial\beta}{\partial s_2}+\frac{\partial\beta}{\partial s_3}$$

where $\partial/\partial s_3=(\boldsymbol{n}_1\times\boldsymbol{n}_2)\cdot\nabla$. By virtue of the Meusnier theorem we have

$$\frac{\partial\beta}{\partial s_3}=N\frac{\sin\beta}{y}\quad\Rightarrow\quad \mathrm{div}\boldsymbol{n}_1=\frac{\partial\beta}{\partial s_2}+N\frac{\sin\beta}{y}\qquad(7)$$

By $M=V/a$ denote the Mach number. It follows from eq.(1') that at $S=S_0$, $I_0=I_{00}$

$$\frac{\partial\ln\rho V}{\partial s_1}=\frac{\partial\ln\rho}{\partial s_1}+\frac{\partial\ln V}{\partial s_1}=\frac{1}{a^2}\frac{\partial I}{\partial s_1}+\frac{\partial\ln V}{\partial s_1}=\qquad(8)$$

$$=-\frac{V}{a^2}\frac{\partial V}{\partial s_1}+\frac{\partial\ln V}{\partial s_1}=(1-M^2)\frac{\partial\ln V}{\partial s_1}$$

Using eqs.(7),(8), we obtain

$$\frac{\partial\beta}{\partial s_2}+N\frac{\sin\beta}{y}+(1-M^2)\frac{\partial\ln V}{\partial s_1}=0\qquad(9)$$

Taking the scalar products of the second equation (4) with $\boldsymbol{n}_2$, $\boldsymbol{n}_1$ and bearing in mind that $(\partial\boldsymbol{n}_1/\partial s_1)\cdot\boldsymbol{n}_2=\partial\beta/\partial s_1$, we have

$$\rho V^2\frac{\partial\beta}{\partial s_1}+\frac{\partial p}{\partial s_2}=0\qquad(10)$$

$$\rho V\frac{\partial V}{\partial s_1}+\frac{\partial p}{\partial s_1}=0\qquad(11)$$

Eqs. (9),(10),11) can only be considered locally, as they involve directional rather than partial derivatives. Denoting by $h_{1,2}$ the Lame coefficients, we have

$$\frac{\partial}{\partial s_1}=\frac{1}{h_1}\frac{\partial}{\partial\varphi},\quad\frac{\partial}{\partial s_2}=\frac{1}{h_2}\frac{\partial}{\partial\psi}$$

$$h_1=(x_\varphi^2+y_\varphi^2)^{1/2}=\left|\nabla\varphi\right|^{-1}=V^{-1},\ h_2=(x_\psi^2+y_\psi^2)^{1/2}=\left|\nabla\psi\right|^{-1}=(\rho Vy^N)^{-1}\qquad(12)$$

By virtue of the Theorem 2 we have

$$\tilde p=p(i,S)|_{I_{00},S_0}=p(I_{00}+V^2/2,\ S_0)$$
$$\tilde\rho=\rho(i,S)|_{I_{00},S_0}=\rho(I_{00}+V^2/2,\ S_0)\qquad(13)$$
$$\tilde M=V/a(i,S)|_{iI,S_0}=V/a(I_{00}-V^2/2,\ S_0)$$

It follows from the comparison of the equality $p_\varphi=-p_I|_{I_{00},S_0}VV_\varphi$ with eq.(11) that $p_I|_{I_{00},S_0}=-\tilde\rho$.
Calculating p_ψ, we obtain that

$$p_\psi=-p_I|_{I_{00},S_0},\quad VV_\psi=\tilde\rho VV_\psi$$

Using eqs.(12) we express finally the invariant form of equations of potential flows written in the natural coordinates (φ,ψ)

$$\tilde\rho(V)y^N\frac{\partial\beta}{\partial\psi}+(1-\tilde M^2(V))\frac{\partial\ln V}{\partial\varphi}+\frac{N}{V}\frac{\sin\beta}{y}=0,\quad\frac{\partial\beta}{\partial\varphi}-\tilde\rho(V)y^N\frac{\partial\ln V}{\partial\psi}=0\qquad(14)$$

Here dependence on thermodynamics is determined only by eqs. (13).
Invariance of eq. (14) consists in that that they do not differ formally from those for Clapeiron's gas. However unlike Clapeiron's gas, in which $a\sim\rho^{\gamma-1}$, $\gamma=c_p/c_V=const$, the sound velocity $\tilde a(V)$ cannot be expressed via density $\tilde\rho(V)$. Therefore so called "replacement principle" [9],[11] does not hold in imperfect gases even in potential flows. In other words, relational positioning of stream lines and level lines of velocity depends on constants I_{00},S_0.

Hodograph transformation
In plane-parallel flows eqs.(14) are homogeneous with respect to derivatives, therefore changing $(\varphi,\psi)\to(V,\beta)$ can be made without trouble by using formulas of type

$$\beta_\varphi=\frac{\partial(\beta,\psi)}{\partial(\varphi,\psi)}=\frac{\partial(\beta,\psi)}{\partial(\beta,V)}\cdot\frac{\partial(\beta,V)}{\partial(\varphi,\psi)}=\psi_V\ \frac{\partial(\beta,V)}{\partial(\varphi,\psi)}$$

However for axially symmetrical flows this approach is impossible as obtained expressions cannot be cancelled by the Jacobian $\partial(\beta,V)/\partial(\varphi,\psi)$. Therefore we apply here another approach, which we have used while designing axially symmetrical supersonic aerodynamic tubes [2] and an inlet valve of the piston engine [12].
Let us write equations of stationary potential flows down

$$\mathrm{div}(y^N\tilde\rho(V)\boldsymbol{V})=\frac{\partial}{\partial x}[y^NV\tilde\rho(V)\cos\beta]+\frac{\partial}{\partial y}[y^NV\tilde\rho(V)\sin\beta]=0\qquad(15)$$

$$\mathrm{rot}\boldsymbol{V}=0\ \Rightarrow\ \frac{\partial}{\partial x}[V\sin\beta]-\frac{\partial}{\partial y}[V\cos\beta]=0$$

Differentiating, we obtain

$$y^N\left(R(V)V_x\cos\beta-\sin\beta\beta_x+R(V)V_y\sin\beta+\cos\beta\beta_y\right)=-N\sin\beta\qquad(16)$$
$$V_x\sin\beta+V\cos\beta\beta_x-V_y\cos\beta+V\sin\beta\beta_y=0$$

where $Q(V)=V\tilde\rho(V)$, $R(V)=d\ln Q(V)/dV$.
Let us change places of dependent and independent variables. Denoting by D the Jacobian $\partial(x,y)/\partial(V,\beta)$) we have

$$V_x=\frac{(V,y)}{(x,y)}=\frac{(V,y)}{(V,\beta)}\frac{(V,\beta)}{(x,y)}=\frac{y_\beta}{D},\ V_y=-\frac{(V,x)}{(x,y)}=-\frac{(V,x)}{(V,\beta)}\frac{(V,\beta)}{(x,y)}=-\frac{x_\beta}{D}\qquad(17)$$
$$\beta_x=-\frac{(y,\beta)}{(x,y)}=-\frac{(y,\beta)}{(V,\beta)}\frac{(V,\beta)}{(x,y)}=-\frac{y_V}{D},\ \beta_y=\frac{(x,\beta)}{(x,y)}=\frac{(x,\beta)}{(V,\beta)}\frac{(V,\beta)}{(x,y)}=\frac{x_V}{D}$$

Substituting formulas (16) into eqs.(15) we obtain

$$R(V)y_\beta\cos\beta+\sin\beta y_V-R(V)x_\beta\sin\beta+\cos\beta x_V=-y^{-N}ND\sin\beta$$
$$y_\beta\sin\beta-V\cos\beta y_V+x_\beta\cos\beta+V\sin\beta x_V=0$$

It follows from first eq. (15) that

$$\psi_V=\psi_xx_V+\psi_yy_V=-y^NQ(V)(x_V\sin\beta-y_V\cos\beta)\qquad(18)$$
$$\psi_\beta=\psi_xx_\beta+\psi_yy_\beta=-y^NQ(V)(x_\beta\sin\beta-y_\beta\cos\beta)$$

Resolving system (6),(7) with respect to derivatives x_V,y_V,x_β,y_β, we have

$$Q(V)y^Nx_V=-\sin\beta\psi_V-R(V)\cos\beta\psi_\beta-Q(V)D\sin\beta\cos\beta$$
$$Q(V)y^Nx_\beta=V\cos\beta\psi_V-\sin\beta\psi_\beta\qquad(19)$$
$$Q(V)y^Ny_V=\cos\beta\psi_V-R(V)\sin\beta\psi_\beta-Q(V)D\sin^2\beta$$
$$Q(V)y^Ny_\beta=\cos\beta\psi_\beta+V\sin\beta\psi_V$$

Cross-differentiating the expressions for yy_V and yy_β, we obtain the equation for the stream function $\psi(V,\beta)$ with the Chaplygin operator on its left-hand side

$$V\psi_{VV}+\left(2-VR(V)\right)\psi_V+R(V)\psi_{\beta\beta}=-Q(2D\cos\beta+D_\beta\sin\beta)N\qquad(20)$$

The Jacobian D is also expressed via derivatives ψ_V and ψ_β. Indeed, substituting expressions (18) into equality $D=x_Vy_\beta-x_\beta y_V$, we obtain

$$D=\frac{V\psi_V^2+R\psi_\beta^2}{Qy^2+\sin\beta\psi_\beta^2}QyN$$

Keeping in mind that

$$Q(V) = V\tilde{\rho}(V)$$
$$R(V) = d\ln Q(V)/dV = d\ln\tilde{\rho}(V)/dV + V^{-1} =$$
$$= -V/a^2(V) + V^{-1} = (1-\tilde{M}^2(V))/V$$

we transform eq.(20) to the final form

$$\psi_{VV} + \psi_V \frac{1+\tilde{M}^2(V)}{V} + \psi_{\beta\beta}\frac{1-\tilde{M}^2(V)}{V^2} = \tag{21}$$
$$= -ND(2\cos\beta + (\ln D)_\beta \sin\beta)\tilde{\rho}(V)$$

Here $\psi = \psi(V,\beta;\ \tilde{\rho}(V), \tilde{M}(V))$, where $\tilde{\rho}(V)$, $\tilde{M}(V)$ are determined by eq.(13). We have

$$D = \frac{V^2\psi_V^2 + (1-\tilde{M}^2(V))\psi_\beta^2}{\tilde{\rho}(V)Vy^2 + \sin\beta\psi_\beta^2}\tilde{\rho}(V)\,yN \tag{22}$$

$$(\ln D)_\beta = 2\frac{V\psi_V\psi_{V\beta} + (1-\tilde{M}^2(V))\psi_\beta\psi_{\beta\beta}/V}{V\psi_V^2 + (1-\tilde{M}^2(V))\psi_{\beta\beta}/V} + \frac{\psi_V V\sin\beta + \psi_\beta\cos\beta}{y} -$$
$$-2\frac{(\psi_\beta\cos\beta + \psi_V V\sin\beta)^2/(V\rho(V)) + \psi_\beta\psi_{\beta\beta}\sin\beta + \psi_\beta^2\cos\beta}{\tilde{\rho}(V)Vy^2 + \psi_\beta^2\sin\beta} \tag{23}$$

The problem of aerodynamic designing is formulated as a boundary problem for eqs.(21)-(23) in some domain in the hodograph plane. Integrating expressions (19) for $x_V, x_\beta, y_V, y_\beta$, we carry out the mapping into physical plane. Arbitrariness of the flow domain allows to optimize device being designed.

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